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# Evolutionary Boussinesq model with nonmonotone friction and heat flux boundary conditions☆

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A B S T R A C T

In this paper we prove the existence and regularity of a solution to a two-dimensional system of evolutionary hemivariational inequalities which describes the Boussinesq model with nonmonotone friction and heat flux. We use the time retardation and regularization technique, combined with a regularized Galerkin method, and recent results from the theory of hemivariational inequalities.



## 1. Introduction

The Boussinesq system of hydrodynamics equations arises from several physical problems when the fluid varies in temperature from one place to another, and we simultaneously observe the flow of fluid and heat transfer. The system couples incompressible Navier–Stokes equations for the fluid velocity and the thermodynamic equation for the temperature distribution. For the derivation of the Boussinesq equations, see [1–3]. The mathematical theory of Navier–Stokes equations has been of the strong interest in the mathematical community for many years, the basic references are monographs [4,5]. The coupled system of incompressible Navier–Stokes problem with the heat equation has been studied in both static and evolutionary cases by many authors, e.g. see [6–9] and the references therein.

In the present paper we study the Boussinesq system which consists with two evolutionary partial differential equations of parabolic type. We impose mixed nonmonotone subdifferential boundary conditions.

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More precisely, we divide the boundary into two parts. On one part the usual Dirichlet condition applies. On the other part, called the contact boundary, we consider a nonmonotone friction law for the velocity, as well as a nonmonotone law for the heat flux. Because of these nonmonotone conditions, the formulation of the problem based on a variational inequality approach and the notion of convex subdifferential cannot be applied. It is worth noting that the subdifferential boundary conditions for Navier–Stokes equations in the convex case have been studied in [10,11] and more recently in [12]. These authors consider multivalued boundary conditions generated by the subdifferential of the norm function. Our work generalizes some of the aforementioned results to the problems with boundary conditions described by the Clarke subdifferential of locally Lipschitz functions, cf. [13]. For this reason, we use the theory of hemivariational inequalities to derive the weak formulation of the problem. For the mathematical theory of hemivariational inequalities modeling stationary (time-independent) problems, we refer to [14–17] and the references therein. For the evolutionary hemivariational inequalities and their various applications to mechanics, we refer the reader to [18–20] and the recent monograph [21].

Furthermore, in our approach we introduce a strong coupling between the Navier–Stokes equations and the heat equation. This coupling together with an additional presence of nonmonotone contact conditions represents the main difficulty of the system under consideration. Note that the Navier–Stokes equations and Stokes problems with nonmonotone boundary conditions and without such coupling have been studied in [22–24]. Finally, we mention that the main tools in the present paper are abstract results from the theory of hemivariational inequalities, cf. [18] and the time retardation method. The latter technique has been successfully applied to coupled systems in viscoelastic damage, as well as to Stefan problem and thermistors. The time retardation method allowed to obtain interesting existence results in [25–27].

The structure of this paper is as follows. In Section 2, we present the preliminary material used later. Section 3 describes the physical setting and the classical formulation of the Boussinesq problem. Section 4 contains the variational formulation of the problem and the proof of our main result on existence and regularity of the solution.

## 2. Preliminaries

In this section we recall definitions and notations used throughout the paper.

We first recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $\varphi\colon X \to \mathbb{R}$, where $(X, \|\cdot\|_X)$ is a Banach space (see [13]). The generalized directional derivative of $\varphi$ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x;v) = \limsup_{y\to x,\ t\downarrow 0} \frac{\varphi(y+tv)-\varphi(y)}{t}.$$

The generalized gradient of $\varphi$ at $x$, denoted by $\partial\varphi(x)$, is a subset of a dual space $X^*$ given by $\partial\varphi(x) = \{\zeta \in X^* \mid \varphi^0(x;v) \geq \langle \zeta, v\rangle_{X^*\times X}$ for all $v \in X\}$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\Gamma$ consisting of two open disjoint sets $\Gamma_1$ and $\Gamma_0$ such that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$. For a vector $\xi \in \mathbb{R}^2$, we denote by $\xi_\nu$ and $\xi_\tau$ its normal and tangential components on the boundary, i.e., $\xi_\nu = \xi \cdot \nu$ and $\xi_\tau = \xi - \xi_\nu \nu$, where dot denotes the inner product in $\mathbb{R}^2$ and $\nu$ is the outward unit normal vector to $\Gamma$.

We introduce the following function spaces

$$E_0 = \{v \in H^1(\Omega)^2 \mid v = 0 \text{ on } \Gamma_0,\ v_\nu = 0 \text{ on } \Gamma_1\}, \qquad V = \{\theta \in H^1(\Omega) \mid \theta = 0 \text{ on } \Gamma_0\}.$$

Moreover, we introduce divergence-free spaces $H^1_\sigma(\Omega)^2 = \{v \in H^1(\Omega)^2 \mid \mathrm{div} v = 0\}$ and

$$E = E_0 \cap H^1_\sigma(\Omega)^2.$$