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# Existence of solutions for $2\,n$th-order nonlinear $p$-Laplacian differential equations

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ABSTRACT

The aim of this paper is to study the existence and multiplicity of weak and classic solutions for a $2\,n$th-order differential equation involving the $p$-Laplacian coupled with periodic boundary conditions. The results are proved by using the minimization argument and an extended Clark's theorem. Some particular cases are shown.



## 1. Introduction

The aim of this paper is to study a $2\,n$th-order differential equation involving the $p$-Laplacian, with periodic boundary conditions. This kind of problems have been widely studied in the literature. For instance, in [1] it is considered a second order problem, in [2] it is obtained the existence of anti-periodic solutions for a $n$th-order problem. Moreover, in [3] it is studied a fourth order problem involving the $p$-Laplacian with deviating terms.

In this paper, we generalize the results obtained in [4] for a second order problem.

For $p > 1$, let us introduce the function $\varphi_p\colon \mathbb{R} \to \mathbb{R}$, defined by:

$$\varphi_p(t) = \begin{cases} t|t|^{p-2} & t \neq 0, \\ 0 & t = 0, \end{cases}$$

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Let us consider the following problem:

$$\left(\varphi_p\left(u^{(n)}(t)\right)\right)^{(n)}+\sum_{i=1}^{n-1}(-1)^i a_i\left(\varphi_p\left(u^{(n-i)}(t)\right)\right)^{(n-i)}+(-1)^n\left(f(t,u(t))-h(t,u(t))\right)=0,\quad t\in[0,T] \tag{1}$$

coupled with the boundary conditions

$$u(T)-u(0)=\cdots=u^{(2\,n-1)}(T)-u^{(2\,n-1)}(0)=0, \tag{2}$$

where $T\geq 0$ and $a_i\geq 0$ for $i=1,\ldots,n-1$.

We introduce the following Banach space:

$$X_p=\left\{u\in W^{n,p}\left(0,T\right)\mid u(T)-u(0)=\cdots=u^{(n-1)}(T)-u^{(n-1)}(0)=0\right\}, \tag{3}$$

where $W^{n,p}\left(0,T\right)$ is the Sobolev space:

$$W^{n,p}\left(0,T\right)=\left\{u\in L^p(0,T):\|u\|_p=\left(\sum_{i=0}^{n}\int_0^T\left|u^{(i)}(t)\right|^p\,dt\right)^{1/p}<\infty\right\}.$$

The function $u\in C^n([0,T])$ is said to be a classical solution of this problem if $\varphi_p\left(u^{(n)}(\cdot)\right)\in C^n([0,T])$ and it satisfies Eq. (1) for $t\in(0,T)$ and periodic conditions (2).

**Remark 1.1.** Realize that, from the derivative chain rule, the assumption $\varphi_p\left(u^{(n)}(\cdot)\right)\in C^n([0,T])$ implies that $u^{(n)}\in C^n([0,T])$. Moreover, if $u^{(n)}$ is not of constant sign on $[0,T]$, in order to have $\varphi_p\left(u^{(n)}(\cdot)\right)\in C^n([0,T])$, we should ask for $\varphi_p\in C^n(\mathbb{R})$.

So, in particular, $u\in C^{(2\,n)}([0,T])$. However, we need to study the regularity of the $p$-Laplacian to ensure that a function which verifies $u\in C^{(2\,n)}([0,T])$ also satisfies $\varphi_p\left(u^{(n)}(\cdot)\right)\in C^n([0,T])$.

For instance, let us consider $n=2$ and $p=3$, for $u\in C^4([0,T])$ we have

$$\left(\varphi_3\left(u''(t)\right)\right)'=\varphi_3'\left(u''(t)\right)\,u^{(3)}(t)=|u''(t)|\,u^{(3)}(t),$$

hence $\varphi_3\left(u''(\cdot)\right)\in C^1([0,T])$, but $\varphi_3\left(u''(\cdot)\right)\notin C^2([0,T])$ if $u''$ is not of constant sign on $[0,T]$ even if $u^{(3)}\in C^1([0,T])$.

The function $u\in X_p$ is said to be a weak solution of (1)–(2) if for every $v\in X_p$ it is verified the following equality:

$$\int_0^T\varphi_p\left(u^{(n)}(t)\right)v^{(n)}(t)\,dt+\sum_{i=1}^{n-1}a_i\int_0^T\varphi\left(u^{(n-i)}(t)\right)\,v^{(n-i)}(t)\,dt$$
$$+\int_0^T\left(f(t,u(t))-h(t,u(t))\right)\,v(t)\,dt=0. \tag{4}$$

The aim of this paper is to ensure the existence of multiple weak solutions of (1)–(2). That is, we look for $u\in X_p$ such that (4) is verified. Then, we ensure that in some cases the weak solution is also a classical solution.

Now, we introduce a condition that $f$ and $h$ must satisfy. Let us consider the following functions:

$$F(t,u)=\int_0^u f(t,s)\,ds,\qquad H(t,u)=\int_0^u h(t,s)\,ds.$$