Nonlinear Analysis: Real World Applications 31 (2016) 23–37

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Attractors for nonclassical diffusion equations with arbitrary polynomial growth nonlinearity

Yongqin Xie[a,*], Qingsong Li[a], Kaixuan Zhu[b]

[a] *School of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha, 410114, PR China*

[b] *School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China*

ARTICLE INFO

Article history:
Received 17 May 2015
Received in revised form 7 January 2016
Accepted 9 January 2016

Keywords:
Asymptotic contractive semigroup
Nonclassical diffusion equation
Global attractor
The whole space
Polynomial growth of arbitrary order

ABSTRACT

In this paper, using a new method (or technique) called "Asymptotic Contractive Semigroup Method" (see Theorem 2.3) we prove the existence of global attractor for the nonclassical diffusion equations in $H^1(\mathbb{R}^n)$. The nonlinearity satisfies the arbitrary order polynomial growth conditions and the forcing term g only belongs to $H^{-1}(\mathbb{R}^n)$. The results obtained essentially improve and complement in some previously published papers.

1. Introduction

In this paper, we investigate the asymptotic behavior of solutions to the following nonclassical diffusion equations:

$$\begin{cases} u_t - \nu\Delta u_t - \Delta u + \lambda u + f(x,u) = g \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where ν, λ are two positive constants, and $g \in H^{-1}(\mathbb{R}^n)$. This equation is called the nonclassical diffusion equation when $\nu > 0$, and when $\nu = 0$, it turns out to be the classical reaction–diffusion equation.

Eq. (1.1) arises in many different areas of mathematics and physics. This equation appears in fluid mechanics, solid mechanics and heat conduction theory [1–6] and the references therein. Eq. (1.1) with a one time derivative appearing in the highest order term ($\nu\Delta u_t$) are called Pseudo-Parabolic or Sobolev–Galpern equations [7–9]. Aifantis [1] proposed a general frame for establishing the equations.

* Corresponding author.
E-mail address: xieyq@csust.edu.cn (Y. Xie).

http://dx.doi.org/10.1016/j.nonrwa.2016.01.004

The long-time behavior of the solutions of Eq. (1.1) has been considered by many researchers; see, e.g. [10–15] and the references therein. In [15], the author has proved the existence of global attractor in $H_0^1(\Omega)$, when $g(x) \in L^2(\Omega)$ under the assumptions that f satisfies critical exponent growth condition corresponding to $N = 3$. Recently, the authors in [12] obtained the existence of a global attractor for $g(x) \in H^{-1}(\Omega)$ only under critical nonlinearity; and when $g(x,t) \in L_b^2(\mathbb{R}, L^2(\Omega))$, they proved the asymptotic regularity and the existence of (non-autonomous) exponential attractor. In [14], the author has obtained the existence and regularity of global attractor for Eq. (1.1) with critical nonlinearity in $H_0^1(\Omega)$. In [16–18], they proved the non-autonomous dynamical system with critical nonlinearity generated by this class of solutions has a pullback attractors or uniform attractors.

The study of the existence of attractors of Eqs. (1.1) as nonlinearity satisfies arbitrary order polynomial growth condition has received considerably less attention in the literature. In this case, some conclusions can be found in to Sun [19] and Ma [20]. However, as the author points out in [12] that these are some mistakes in the coauthor's earlier paper [19]. The methods presented in [21] cannot be used in this equation. In [20], they proved the existence of global attractor in $H^1(\mathbb{R}^n)$ with the initial data $u_0 \in H^2(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^2)$(see [20, Lemma 4.5]).

As we know, if we want to prove the existence of global attractors, the key point is to obtain the compactness of the semigroup in some sense. Since Eq. (1.1) contains the term Δu_t, it is different from the usual reaction–diffusion equation essentially. For example, the solutions of usual reaction–diffusion equations have some smoothing effect. However, for Eq. (1.1), if the initial data u_0 belongs to $H^1(\mathbb{R}^n)$, the solutions $u(x,t)$ of Eq. (1.1) are always in $H^1(\mathbb{R}^n)$, and have no higher regularity, which is similar to the hyperbolic equations. The nonlinearity satisfies polynomial growth of arbitrary order condition, so Sobolev compact embedding theorems are no longer useful. The asymptotic compactness of solutions cannot be obtained by the usual method. Furthermore, the embedding $H^1(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is no longer compact. We cannot obtain the precompactness of sequence in $L^2(0,T;L^2(\mathbb{R}^n))$ by the boundedness in $L^2(0,T;H^1(\mathbb{R}^n))$.

In [22,23], the authors prove asymptotic compactness for the weakly damped wave-type equations by using contractive function method, in which need the conclusion that the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, $1 \le p < 5$ is compact. However, what we consider is supercritical exponents and the embedding is not compact that we cannot obtain the compactness, that is to say, we cannot obtain the contractive function in [23] directly.

In this paper, the main contribution of this paper is to extend the contractive function method in [23]. The first the conception, Asymptotic Contractible Semigroup, is defined, and a new priori estimates for the existence of global attractors in unbounded domains is provided. Then we obtain the existence of a global attractor for Eq. (1.1) with a polynomial growth nonlinearity in $H^1(\mathbb{R}^n)$.

For nonlinearity, we always assume that $f(x,s) = f_1(s) + a(x)f_2(s)$ satisfies:

(H1) $\alpha_1|s|^p - \beta_1|s|^2 \le f_1(s)s \le \gamma_1|s|^p - \delta_1|s|^2$,

$$p \ge 2, f_1(s)s \ge 0, \qquad f_1'(s) \ge -\mu_0 \quad \text{and} \quad \beta_1 < \lambda;$$

(H2) $\alpha_2|s|^p - \beta_2 \le f_2(s)s \le \gamma_2|s|^p - \delta_2, p \ge 2$, and $f_2'(s) \ge -\mu_0$;
(H3) $a(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), a(x) > 0$,

for any $s \in \mathbb{R}$, where $\alpha_i, \beta_i,\ \gamma_i,\ \delta_i (i = 1,2)$ and μ_0 are all positive constants.

For conveniences, hereafter let $|u|$ be the modular (or absolute value) of u and $|\cdot|_p$ be the norm of $L^p(\mathbb{R}^n)(p \ge 1)$ and $\|\cdot\|_1 = \left(|\cdot|_2^2 + |\nabla \cdot|_2^2\right)^{1/2}$ be the norm of $H^1(\mathbb{R}^n)$, $(\cdot,\cdot)$ be the inner product of $L^2(\mathbb{R}^n)$. Let X be a Banach space, and $u \in X, v \in X^\star$, $X^\star$ is the dual space of X,

$$\langle u, v\rangle = \int_{\mathbb{R}^n} uv.$$