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Singular solutions for a class of traveling wave equations arising in hydrodynamics

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ABSTRACT

We give an exhaustive characterization of singular weak solutions for ordinary differential equations of the form $\ddot{u}\,u + \frac{1}{2}\dot{u}^2 + F'(u) = 0$, where F is an analytic function. Our motivation stems from the fact that in the context of hydrodynamics several prominent equations are reducible to an equation of this form upon passing to a moving frame. We construct peaked and cusped waves, fronts with finite-time decay and compact solitary waves. We prove that one cannot obtain peaked and compactly supported traveling waves for the same equation. In particular, a peaked traveling wave cannot have compact support and vice versa. To exemplify the approach we apply our results to the Camassa–Holm equation and the equation for surface waves of moderate amplitude, and show how the different types of singular solutions can be obtained varying the energy level of the corresponding planar Hamiltonian systems.

1. Introduction

In the present paper we propose to study certain types of weak solutions for ordinary differential equations (ODE) of the form

$$\ddot{u}\,u + \frac{1}{2}\dot{u}^2 + F'(u) = 0, \tag{1}$$

where F is an analytic function. Our motivation stems from the fact that a variety of model equations arising in the context of hydrodynamics, among them the well-known Camassa–Holm equation (cf. [1–3]) and the related equation for surface waves of moderate amplitude (cf. [4–8]), are reducible to an ODE of the form (1) upon passing to a moving frame. Owing to the fact that every solution of Eq. (1) may be interpreted as

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a traveling wave of a suitable underlying partial differential equation (PDE) we will call the solutions of (1) traveling waves.

The singular nature of Eq. (1) accounts for the non-uniqueness of certain solutions, which we call *singular solutions*. These are in general weak solutions, but have stronger regularity than one would expect a priori: the solutions are analytic except for a countable number of points at which the equation is satisfied in the limit. Furthermore, Eq. (1) admits an order reduction which allows us to see that under certain conditions on F, the solutions are actually classical solutions of this reduced equation.

The main result of this paper consists in the exhaustive characterization of singular solutions of (1) from qualitative properties of the function $F(u)$. We show that Eq. (1) admits solutions with peaks and cusps, fronts with finite-time decay and solitary solutions with compact support. Furthermore, we find that one cannot obtain peaked and compactly supported solutions for the same F. In particular, a peaked solution cannot have compact support and vice versa. The characterization of classical solutions of (1) will be covered only very briefly for the convenience of the reader, since our main focus lies in the analysis of singular solutions.

We apply our results to the aforementioned nonlinear partial differential equations, and show how the different types of singular solutions are obtained varying the energy levels of the Hamiltonian planar differential system corresponding to (1). It lies beyond the scope of this paper to prove in full generality that every weak solution of (1) is also a weak traveling wave solution of an underlying PDE. For a discussion of this problem we refer the reader to [9,10], where it is shown that in the special case of the Camassa–Holm equation every weak solution of (1) is a weak traveling wave solution of the underlying PDE. Following similar steps the same result can be shown for the equation of surface waves of moderate amplitude.

The structure of the paper is as follows. In Section 2, we give the precise definitions of weak and singular solutions and provide a preliminary result on the non-uniqueness of solutions of (1). In Section 3 we introduce the notion of *elementary forms*, classical solutions of (1) defined on a subset of $\mathbb{R}$, from which we construct singular solutions. Furthermore, we discuss how the qualitative features of any traveling wave solution can be obtained from the properties of F. The main results of the paper, Propositions 6, 7, 11 and 12 are presented in Section 4 which is devoted to the complete characterization of singular solutions. In Section 5, we characterize the classical and singular traveling waves of the Camassa–Holm equation and the equation for surface waves of moderate amplitude in shallow water.

2. Weak and singular solutions

Our focus lies in the characterization of solutions which are not classical, so we require a weak formulation of (1). Keeping in mind that any solution of (1) can be interpreted as a traveling wave of an underlying PDE, we will consider only bounded solutions.

Definition 1. We say that a bounded function $u \in H^1_{loc}(\mathbb{R})$ is a *traveling wave solution (TWS)* if it satisfies (1) in the sense of distributions, i.e. if u satisfies

$$\int_{\mathbb{R}} (u^2)_t \phi_t + (u_t)^2 \phi - 2F'(u)\phi \, dt = 0, \tag{2}$$

for any test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. We say that u is a *strong TWS* if it satisfies (1) in the classical sense.

It turns out that the concept of weak solutions is quite crude. Indeed, if no further conditions are imposed it is possible to find a plethora of weak solutions of (1) giving rise to TWS with very complex shapes. For instance it is known that the Camassa–Holm equation can have TWS of the form $u = \varphi(t)$ such that some of its level sets $\{\varphi(t) = k\}$ are cantor sets, cf. [10]. In the present paper, however, we are interested in those solutions which fail to be strong TWS because of the singularity, but which still have a certain degree