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The pressure distribution in extreme Stokes waves

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ABSTRACT

In this paper we prove that the pressure beneath an extreme Stokes wave over finite depth is strictly increasing with depth. Additionally it is shown that the pressure decreases in moving between a crest-line and trough-line, while it is stationary with respect to the horizontal coordinate along these lines themselves.

1. Introduction

In the current paper we investigate the behaviour of the pressure profile induced by a Stokes wave of greatest height propagating in a fluid body of finite-depth. A Stokes waves of greatest height, also called an extreme Stokes waves, is a steady, periodic, two-dimensional flow, characterised by the appearance of a stagnation point on the free surface, specifically at the wave crest, which appear as cusps with an included angle of 120° cf. [1]. The symmetries and regularity of steady periodic waves have been well established under a variety of conditions, including gravity waves incorporating vorticity cf. [2–5]. However the presence of stagnation points in the flow introduces a number of mathematical complications related to the regularity of the free surface [6,7], which require careful and subtle treatment when analysing the governing equations of the flows. The analytical techniques developed in this paper by means of an excision of the stagnation points allow us to deduce several key features of the pressure in the fluid flow. Specifically it is shown that the pressure decreases as we move horizontally between a crest-line and a trough-line, while along these lines the pressure is stationary with respect to the horizontal coordinate. Secondly it will be shown that the pressure strictly increases with depth at any point throughout the fluid domain.

Incompressibility and irrotationality of regular Stokes waves allow one to construct a conformal map of the fluid domain with analytic continuation to the boundary, the so called hodograph transform. Using the hodograph transform one may equivalently reformulate the free-boundary problem governing two-dimensional Stokes waves as a fixed-boundary problem on the conformal image of the fluid domain, wherein

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one may readily employ several powerful maximum principles. However the presence of a stagnation point in an extreme Stokes wave ensures that the hodograph transform does not have analytic continuation to the boundary; as such the fixed-boundary and free-boundary problems may no longer be identified on the boundary. Moreover, owing to the lack of regularity of the free surface, the boundary of the conformal domain lacks the necessary regularity to impose maximum principles. Nevertheless by excising the stagnation point from the conformal domain, we restore the necessary regularity of the boundary to impose maximum principles and also ensure the hodograph transform is conformal throughout. Then by means of a uniform limiting process cf. [8] we may extend the application of maximum principles from the excised domain to the fluid domain. The uniform limiting process was used to great effect in the work [8], which proved the existence of extreme deep-water Stokes incorporating cusps with an included angle of 120°. Moreover, the same technique was employed in the work [9] to prove the convexity of the surface profile of Stokes waves of greatest height. In the recent works [6,10], the uniform limiting process was effectively used to show that particles travelling on the streamlines of extremes Stokes waves, in finite and infinite depth, do not form closed trajectories.

Extending the methods developed in [6,10], we demonstrate the aforementioned qualitative behaviour of the pressure in an extreme Stokes wave in fluid body of finite depth; see [11–13] for recent results concerning the pressure in deep-water Stokes waves. Pressure measurements within a water body are of significant importance both experimentally and theoretically. In many respects the pressure determines several physical features of the fluid body, for example governing the trajectories followed by individual fluid particles, see [14,6,15,16,11,10] for some recent results on particle trajectories in regular and extreme Stokes waves. From a practical point of view, wave measurements can be extremely difficult to execute [17,18] and a widely used method of obtaining information is through pressure data [19,20]. These pressure measurements convey information about the surface profile of the water wave by means of the pressure transfer function c.f. [21–25] for some recent results in this regard. From an experimental and commercial perspective, information about the velocity and pressure beneath surface waves are crucial to the design of cost effective off-shore structures [26].

In Section 2 we introduce the free boundary problem for extreme Stokes waves in the moving frame, that is to say the frame which moves with wave phase speed of the two-dimensional flow. Section 3 examines the stream function and velocity potential for the flow, which follow from the incompressibility and irrotationality of the flow, and introduce the hodograph transform defined in terms of these functions. The free boundary problem is reformulated as a fixed boundary problem on the conformal image of the fluid domain under the hodograph transform. In Section 4 we prove the main theorem of this paper. Theorem 4.1 states that the fluid pressure is strictly decreasing as a function of the horizontal coordinate x between a crest-line and subsequent trough-line, except along these lines themselves, where the pressure is stationary with respect to this x-coordinate. Additionally it is proven that the pressure is a strictly decreasing function of the vertical coordinate y, that is the pressure increases with the fluid depth.

2. Physical considerations

2.1. The free boundary problem

There are several structural features common to all Stokes waves, which are crucial for the analysis of the flow. Stokes waves are steady, irrotational and periodic wave trains, moving with fixed wave speed c relative to the physical frame with coordinates (X, Y, Z). In a Stokes wave of greatest height, the free surface profile $\eta := \eta(X - ct)$ is symmetric about any crest-line and moreover is strictly decreasing between wave-crest and wave-trough cf. [7]. In general for Stokes waves, the horizontal fluid velocity u and pressure P are periodic and symmetric about the crest-line while the vertical velocity v is periodic and antisymmetric about the