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A minimization problem for an elliptic eigenvalue problem with nonlinear dependence on the eigenparameter

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ABSTRACT

In this paper we examine an eigenvalue optimization problem. Given two nonlinear functions $\alpha(\lambda)$ and $\beta(\lambda)$, find a subset D of the unit ball of measure A for which the first Dirichlet eigenvalue of the operator $-\mathrm{div}((\alpha(\lambda)\chi_D+\beta(\lambda)\chi_{D^c})\nabla u)=\lambda u$ is as small as possible. This sort of nonlinear eigenvalue problems arises in the study of some quantum dots taking into account an electron effective mass. We establish the existence of a solution, and we propose a numerical algorithm to obtain an approximate description of the optimizer.

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1. Introduction

Let Ω be a bounded, connected, open set in $\mathbb{R}^N$ with smooth boundary. Assume that A is a given positive number, $0<A<|\Omega|$, where $|.|$ denotes the Lebesgue measure. Given a measurable set $D\subset\Omega$ with $|D|=A$ and continuous positive functions $\alpha(\lambda)$ and $\beta(\lambda)$ for $\lambda\geq 0$, consider the following nonlinear eigenvalue problem

$$-\mathrm{div}(G(\lambda,x)\nabla u)=\lambda u \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $G(\lambda,x)=\alpha(\lambda)\chi_D+\beta(\lambda)\chi_{D^c}$, $\alpha(\lambda)\geq\beta(\lambda)$. In this paper, λ is the principal eigenvalue (ground state energy) or the smallest positive eigenvalue of (1.1) and $u=u(x)$ is a corresponding eigenfunction.

We are interested in the cases where $\alpha(\lambda)$ and $\beta(\lambda)$ are nonlinear functions of the parameter λ. Indeed, Eq. (1.1) can be regarded as a nonlinear elliptic eigenvalue problem such that the nonlinearity originates from the nonlinear dependence on the eigenparameter.

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Such nonlinear eigenvalue problems appear as the Hamiltonian equation governing some quantum dot nanostructures, where $\alpha(\lambda)$ and $\beta(\lambda)$ correspond to the effective mass of the carrier (electron or hole) and the surrounding matrix, respectively, and λ is the ground state energy [1–4]. A real physical phenomenon modeled by Eq. (1.1) is the heterostructures of different semiconductors where the electron effective mass depends on both the energy and position [1,2].

It is known that the ground state energy of (1.1) depends on the set D, the region with effective mass α, and we use the notation $\lambda(D)$ as we want to emphasize this dependence. To determine the system's profile which gives the minimum ground state energy, we study the following optimization problem

$$\inf_{\substack{D\subset\Omega\\ |D|=A}} \lambda(D). \tag{1.2}$$

Solutions of minimization problem (1.2) correspond to the physical systems which have lowest ground state energies and so are most stable structures from a physical point of view. Regarding the importance of stable structures in designing electronic and photonic devices based on quantum dots, it would be interesting to find them.

Problem (1.1) is in fact a generalization of the linear case $G(\lambda, x) = \alpha\chi_D(x) + \beta\chi_{D^c}(x)$ where α and β are two positive constants, $\alpha > \beta$. In the linear case, the optimization problem (1.2) is the problem of optimal design where two material phases are to be distributed inside a fixed region Ω. This problem is known to often have no solution other than microstructural designs [5]. However, the original problem (1.2) may still have a solution for specific Ω for instance when Ω is a ball. Existence of a radially symmetric optimal set for the linear case has been established in [6] when $\Omega = \mathcal{B}(0, R)$ is a ball with radius R centered at the origin. Conca et al. have revived interest in this problem by giving a new simpler proof of the existence result only using rearrangement techniques [7]. For the one-dimensional case, Krein has shown in [8] that the unique minimizer is a ball $\mathcal{B}(0, R^*)$. This suggests for higher dimensions that $\mathcal{B}(0, R^*)$ is a natural candidate to be the optimal domain. This conjecture has been supported by numerical tests in [9]. In addition, it has been shown in [10] employing second order shape derivative calculus that $\mathcal{B}(0, R^*)$ is a local minimum for the optimization problem when A is small enough. In spite of the above evidences, it has been established in [11] that the conjecture is not true at least in two- or three-dimensional spaces when α and β are close to each other (low contrast regime) and A is sufficiently large. This makes clear that the optimal domain cannot be a ball. The theoretical base for the result is an asymptotic expansion of the eigenvalue with respect to $\alpha - \beta$ as $\alpha \to \beta$, which allows one to approximate the optimization problem by a simple minimization problem. Based upon the properties of Bessel functions, it has been proved in [12] that the conjecture is not true not only for two- or three-dimensional spaces, but also for all dimensions $n \geq 2$. Recently, Laurian has proved that the optimal domain in low contrast regime is either a centered ball or the union of a centered ball and a centered ring touching the boundary, depending on the prescribed volume ratio between the two materials [13].

In this paper we want to generalize those investigations to the case where $\alpha(\lambda)$ and $\beta(\lambda)$ are nonlinear functions of the parameter λ. First, we will show the existence of a solution for the case where Ω is a ball centered at the origin. Thereafter we will address the question of the configuration of the optimal domain. It will be demonstrated that the optimal set in the one-dimensional problem is a symmetric interval centered at the origin. Despite the conclusion in $\mathbb{R}^1$, a ball is not an optimal shape thanks to findings of the linear cases in two- or three-dimensional space. At last, we need an algorithm that allows us to compute numerical approximations of the optimal solution of the problem (1.2). An efficient and convergent numerical algorithm will be derived based upon the level sets of the gradient of the eigenfunction and the safeguarded iteration.

Let us recall here that nonlinear eigenvalue problems and optimization problems have many applications in engineering and applied sciences and these problems have been intensively attractive to mathematicians in the past decades [14]. However, it should be mentioned that the majority of the investigated nonlinear models are nonlinear in their differential operator part [15–17]. We note that Eq. (1.1) has nonlinear dependence on the parameter λ and such systems have been under less attention in this field of study [18,19].

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