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Analytic integrability inside a family of degenerate centers

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ABSTRACT

In this paper we study the analytic integrability around the origin inside a family of degenerate centers or perturbations of them. For this family analytic integrability does not imply formal orbital equivalence to a Hamiltonian system. It is shown how difficult is the integrability problem even inside this simple family of degenerate centers or perturbations of them.

1. Introduction

This work centers in determining the existence of analytic first integrals in a neighborhood of a degenerate center singular point which indeed is a center or which is a perturbation of a degenerate center. It is well-known that a system has a center at a singular point only if it is monodromic and it has either linear part of center type, i.e. with imaginary eigenvalues (nondegenerate point), or nilpotent linear part (nilpotent point) or null linear part (degenerate point). Any nondegenerate center has always a local analytic first integral in a neighborhood of its singular point, see [1–4]. However, there are nilpotent and degenerate centers without a local analytic first integral, see [5–11,4,12] and references therein. There are methods to detect nondegenerate and nilpotent centers of a given family of polynomial vector field, see [9–11]. However there is no method to detect centers for a general degenerate singular point.

Any nilpotent center has a local analytic first integral if, and only if, it is analytically equivalent to the Hamiltonian system $\dot{x} = y$, $\dot{y} = -x^{2k-1}$ where $k > 1$, see for instance [13]. The integrability problem has been studied for a few families of degenerate singular points.

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In [14] the analytic integrability problem for degenerate systems of the form

$$\dot{x} = y^3 + 3\mu x^2 y + o(|x,y|^3), \qquad \dot{y} = -x^3 - 3\mu x y^2 + o(|x,y|^3), \quad \mu \in \mathbb{R}, \tag{1.1}$$

was analyzed and the following result was established.

Theorem 1.1. *System (1.1) is analytically integrable if, and only if, it is formally equivalent to* $\dot{x} = y^3+3\mu x^2 y$, $\dot{y} = -x^3 - 3\mu x y^2$.

In [15] the analytic integrability problem for degenerate systems of the form

$$\dot{x} = y^3 + 2ax^3 y + \cdots, \qquad \dot{y} = -x^5 - 3ax^2y^2 + \cdots, \quad a \in \mathbb{R}, \tag{1.2}$$

was also studied where here the dots mean terms of higher order than the first component in the quasi-homogeneous expansion (see definition of quasi-homogeneous expansion below). The next result was established in [15].

Theorem 1.2. *System (1.2) is analytically integrable if, and only if, it is formally equivalent to* $\dot{x} = y^3+2ax^3y - 2\beta_9 x^4 y$, $\dot{y} = -x^5 - 3ax^2y^2 + 4\beta_9 x^3 y^2$, *where* β_9 *is written in the parameters of the first quasi-homogeneous components of system (1.2).*

The integrability problem for these two previous families can be solved using the following result given in [13].

Theorem 1.3. *Let us assume that the lowest-degree quasi-homogeneous term of the degenerate system is* $\mathbf{F}_r = \mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$, *where* h *has only simple factors. Then, the quoted system is formally integrable if and only if it is formally conjugated, via dissipative transformations, to a divergence-free system.*

First we give some comments and results about the integrability problem for a vector field $\mathbf{F} = \sum_{j\geq r} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$, in function of its first quasi-homogeneous component $\mathbf{F}_r$. As we have said Theorem 1.3 solves the integrability problem in the case that $\mathbf{F}_r = \mathbf{X}_h$ where all the irreducible factors of h over $\mathbb{C}[x,y]$ are simple. The case $\mathrm{div}\,(\mathbf{F}_r) \neq 0$ with $\mathbf{F}_r$ reducible or $\mathbf{F}_r = \mathbf{X}_h$ where h has multiple factors is not solved and it is an open problem.

However a necessary condition in order that $\mathbf{F}$ be integrable is that $\mathbf{F}_r$ be also integrable. The integrability problem for $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ with $\mathbf{D}_0 = (t_1x, t_2y)^T$, $\mu \neq 0$ is solved in [16], see Theorem 4.12. This theorem shows the necessity of certain resonances in the parameters of the vector field in order that $\mathbf{F}_r$ be integrable. Moreover such resonances determine the exponents of the irreducible factors of h that appear in the first integral.

Lemma 1.4. *Let* $\mathbf{F}_r = (P,Q) \in \mathcal{Q}_r^{\mathbf{t}}$ *irreducible. If* $I \in \mathcal{P}_i^{\mathbf{t}}$ *is a first integral of* $\mathbf{F}_r$, *then* $i \geq r + |\mathbf{t}|$ *and exists* $f \in \mathcal{P}_{i-r-|\mathbf{t}|}^{\mathbf{t}}$ *such that* $f\,\mathbf{F}_r = \mathbf{X}_I$.

Proof. The reasoning is taken from the proof of [16, Theorem 3.1]. We have $\nabla I \cdot \mathbf{F}_r = 0$. As the components of $\mathbf{F}_r$ have not common factors, we have that exist $f \in \mathcal{P}_{i-r-|\mathbf{t}|}^{\mathbf{t}}$ such that $\nabla I = f(-Q,P)$. Thus $\mathbf{X}_I = f(P,Q)^T = f\,\mathbf{F}_r$. ■

On the other hand if $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ is integrable and I is a first integral, from the previous lemma we deduce that exists a quasi-homogeneous function f such that $I = \frac{1}{i}\mathbf{D}_0 \wedge \mathbf{X}_I = \frac{1}{i} f\,\mathbf{D}_0 \wedge \mathbf{F}_r = \frac{r+|\mathbf{t}|}{i} f\,h$. Hence the integrability problem of a quasi-homogeneous vector field with not null divergence is equivalent to the integrability problem of a quasi-homogeneous Hamiltonian vector field where its Hamilton function has multiple factors. It is logical to think that the resonances appearing in a problem are also determinants in the other.