Nonlinear Analysis: Real World Applications 31 (2016) 356–387

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Spatial resonance and Turing–Hopf bifurcations in the Gierer–Meinhardt model

Rui Yang, Yongli Song*

Department of Mathematics, Tongji University, Shanghai 200092, China

HIGHLIGHTS

- Obtain the algorithm of normal form for the spatial resonance bifurcation.
- Study the dynamics of the GM model near the codimension-2 bifurcation point.
- Obtain the existence of multiple spatially inhomogeneous equilibria for GM model.
- Obtain the stable spatially inhomogeneous periodic solutions for GM model.

ARTICLE INFO

Article history:
Received 28 June 2015
Received in revised form 19 February 2016
Accepted 20 February 2016

Keywords:
Gierer–Meinhardt model
Spatial resonance bifurcation
Turing–Hopf bifurcation
Normal form
Dynamical classification

ABSTRACT

Gierer–Meinhardt system as a molecularly plausible model has been proposed to formalize the observation for pattern formation. In this paper, the Gierer–Meinhardt model without the saturating term is considered. By the linear stability analysis, we not only give out the conditions ensuring the stability and Turing instability of the positive equilibrium but also find the parameter values where possible Turing–Hopf and spatial resonance bifurcation can occur. Then we develop the general algorithm for the calculations of normal form associated with codimension-2 spatial resonance bifurcation to better understand the dynamics neighboring of the bifurcating point. The spatial resonance bifurcation reveals the interaction of two steady state solutions with different modes. Numerical simulations are employed to illustrate the theoretical results for both the Turing–Hopf bifurcation and spatial resonance bifurcation. Some expected solutions including stable spatially inhomogeneous periodic solutions and coexisting stable spatially steady state solutions evolve from Turing–Hopf bifurcation and spatial resonance bifurcation respectively.

1. Introduction

In developmental biology, the percussive paper by Turing [1] in 1952 brings us about the concept of Turing instability or diffusion-driven instability. Since then, a variety of phenomena generated by chemical

* Corresponding author.
E-mail address: 05143@tongji.edu.cn (Y. Song).

http://dx.doi.org/10.1016/j.nonrwa.2016.02.006

kinetics have been translated into the challenging mathematical problems, accompanied by different models to describe the biological pattern formation [2]. One of the prototype models for investigating pattern formation in developmental biology is the Gierer–Meinhardt model which has been taken as an example to explain the mechanism to form patterns: short range activation and long range inhibition [3]. Two kinds of morphogenesis serve as activator and inhibitor which oppose the action of each other. The so-called G–M model is:

$$\begin{cases} \dfrac{\partial U(x,t)}{\partial t} = \rho_0\rho + c\rho\dfrac{U^2(x,t)}{V(x,t)} - \theta_1 U(x,t) + d_1\dfrac{\partial^2 U(x,t)}{\partial x^2}, \\ \dfrac{\partial V(x,t)}{\partial t} = c_1\rho_1 U^2(x,t) - \theta_2 V(x,t) + d_2\dfrac{\partial^2 V(x,t)}{\partial x^2}. \end{cases} \tag{1}$$

Here $U(x,t)$ and $V(x,t)$ are the concentrations of the activator and inhibitor; d_1 and d_2 are the diffusion velocities of the activator and inhibitor respectively; $\rho_0\rho$ is the source of activator; θ_1 and θ_2 are the degradation coefficients of the activator and inhibitor while $c\rho$ and $c_1\rho_1$ relate to the productions of both morphogenesis.

For the Gierer–Meinhardt system with saturation term, there are several literatures on it. The stability of the stripe solutions and the effect of saturation on the Gierer–Meinhardt model have been analyzed in a rectangular domain [4]. Chen et al. [5] have obtained a global bifurcation diagram of non-trivial periodic orbits and steady state solutions. Yang et al. [6] have investigated the stability of the trivial and nontrivial equilibria and the existence of Hopf and steady state bifurcations of the Gierer–Meinhardt system with saturation term.

As for system (1), we now briefly summarize the correlative studies. Gierer and Meinhardt focused on the numerical method and have derived a number of patterns relevant to the formation of biological structures in [3]. Ruan [7] has considered the diffusion-driven instability of the homogeneous equilibrium and periodic solution. Some bifurcation analysis and pattern simulations have been carried out by Gonpot et al. in [8]. Ghergu [9] has established various results of existence, regularity and boundary behaviors in the Gierer–Meinhardt system subject to Dirichlet boundary condition. Ghergu [10] has proved the existence of optimal solutions, derived an optimality system, and determined optimal solutions. Zou [11] has established certain sufficient conditions for global existence with a homogeneous Neumann boundary condition. The formations of polar, symmetric and periodic structures from the Gierer–Meinhardt model have been extensively researched: The non-homogeneous steady states of system (1) such as peak and spike solutions have been explored in [12–14]. The existence and stability of wave-like stationary solution have been concerned in [15]. For the semi-discrete form of Gierer–Meinhardt system (1), the conditions of Turing instability have been obtained and different patterns have been shown in [16] and so on.

Liu et al. [17] have considered Hopf bifurcations and steady state bifurcations for the Gierer–Meinhardt system (1) subject to Dirichlet boundary condition in one-dimension space $(0, l\pi)$. In this study, we aim at the spatially codimension-2 bifurcations including Turing–Hopf bifurcation and spatial resonance bifurcation for system (1) with Neumann boundary condition. There are several reasons for choosing Neumann boundary condition. The major one is that the self-organization of patterns is interesting for Neumann boundary condition implying no external input. Wei et al. [18] have investigated a two-species glycolysis model to analyze the steady state bifurcations from a double eigenvalue applying Lyapunov–Schmidt technique and singularity theory. We would also like to mention that the effects of density-dependent nonlinear diffusion on pattern formation in the biochemical systems have recently been investigated in [19,20]. In this paper we not only theoretically obtain the existence of the steady state solution with a new mode but also study the stability of the solution and obtain the bifurcation diagram in the whole parameter plane.

It turns out to be a general principle that the stability of the steady state solution is positively related to the shape of the steady state. That is to say, the more complicated the shape of the steady state is, the less stability the steady state has. When two steady state solutions with different resonant modes interact, there