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The Euler–Lagrange equation for the Anisotropic least gradient problem

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ABSTRACT

In this paper we find the Euler–Lagrange equation for the anisotropic least gradient problem

$$\inf \left\{ \int_\Omega \phi(x, Du) \ : \ u \in BV(\Omega), \ u|_{\partial\Omega} = f \right\}$$

being ϕ a metric integrand and $f \in L^1(\partial\Omega)$. We also characterize the functions of ϕ-least gradient as those whose boundary of the level set is ϕ-area minimizing in Ω.

1. Introduction

In [1], motivated by the Conductivity Imaging Problem, it is studied the following general least gradient problem

$$\inf \left\{ \int_\Omega \phi(x, Du) : u \in BV(\Omega), \ u|_{\partial\Omega} = f \right\} \tag{1.1}$$

where ϕ is a metric integrand and $f \in C(\partial\Omega)$.

Under some restrictions on the metric integrand ϕ and assuming that Ω satisfies a barrier condition, in [1] it is proved that the problem (1.1) has a minimizer for every $f \in C(\partial\Omega)$, which is unique under certain more restrictive assumptions on ϕ.

For the special case $\phi(x, \xi) := g(x)|\xi|$, the problem (1.1) is the weighted least gradient problem:

$$\inf \left\{ \int_\Omega g(x)|Du| : u \in BV(\Omega), \ u|_{\partial\Omega} = f \right\} \tag{1.2}$$

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which appears in [2] in connection with the Conductivity Imaging Problem (see [3] for a good survey about this problem). This problem consists in recovering the isotropic conductivity σ of an object from the knowledge of the magnitude of one current density $|J|$ in the interior. We now briefly describe this problem from the mathematical point of view. Let $\sigma(x)$ a positive function that models the inhomogeneous isotropic conductivity of a body Ω. If u is the electrical potential corresponding to the voltage f on the boundary of Ω, then u solves the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

By Ohm's law, the corresponding current density is $J = -\sigma\nabla u$. By taking the absolute value in Ohm's law and plugging it into the conductivity equation in (1.3), the conductivity imaging problem becomes the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(|J| \dfrac{\nabla u}{|\nabla u|} \right) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The Conductivity Imaging Problem corresponds to the inverse problem of determining σ from the knowledge of $|J|$ inside Ω, with a known f prescribed on $\partial\Omega$. Such internal data can be obtained from Magnetic Resonance Imaging Measurements as shown in [4]. It was first proved in [2] that the corresponding voltage potential u is the unique solution of the weighted least gradient problem:

$$\inf \left\{ \int_\Omega g(x)|\nabla u|, : u \in W^{1,1}(\Omega) \cap C(\overline{\Omega}),\ u|_{\partial\Omega} = f \right\}, \tag{1.5}$$

with $g = |J|$. This result has been extended to $u \in BV(\Omega)$ in [5]. We observe that the Euler–Lagrange equation of the variational problem (1.5) is, formally, the degenerate elliptic equation (1.4). As was point out in [2], due to the degeneracy of (1.5) at the points where the gradient vanishes, the notion of a solution needs to be defined carefully and this is one of the aims of this work.

Biological tissues such as muscle or nerve fibres are known to be electrically anisotropic (see e.g. [6]). In [7] the authors present a method for recovering the conformal factor of an anisotropic conductivity matrix in a known conformal class from one interior measurement. They assume that the matrix-valued conductivity function is of the form

$$\sigma(x) = c(x)\sigma_0(x),$$

with $\sigma_0(x)$ known from the Diffusion Tensor Imaging method (see [8]) and with the cross-property factor $c(x)$ a scalar function to be determined. In [7] the author showed that the corresponding voltage potential u is the unique solution of a general least gradient problem like (1.1), where ϕ is given by

$$\phi(x,\xi) = a(x) \left(\sum_{i,j=1}^N \sigma_0^{ij}(x)\xi_i\xi_j \right)^{\frac{1}{2}}, \tag{1.6}$$

$$a = \sqrt{\sigma_0^{-1} J \cdot J}, \tag{1.7}$$

and J is the current density vector field generated by imposing the voltage f at $\partial\Omega$. Let us remark that in [7] it is assumed that σ_0 satisfies

$$m|\xi|^2 \le \sum_{i,j=1}^N \sigma_0^{ij}(x)\xi_i\xi_j \le M|\xi|^2$$

for some constants $0 < m, M < \infty$ and all $\xi \in R^N$, consequently, we have that the metric integrand ϕ defined by (1.6) satisfies the assumptions we need to impose to get our results.