Nonlinear Analysis: Real World Applications 31 (2016) 552–568

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Global existence and long-time behavior of the initial–boundary value problem for the dissipative Boussinesq equation☆

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A R T I C L E I N F O

Article history:
Received 24 March 2015
Received in revised form 7 November 2015
Accepted 3 March 2016
Available online 31 March 2016

Keywords:
Dissipative Boussinesq equation
Global well-posedness
Global attractor

A B S T R A C T

We investigate the initial–boundary value problem of n-dimensional ($n \geq 1$) dissipative Boussinesq equation. The global existence, uniqueness and stability of the strong solutions are obtained by means of the Faedo–Galërkin method. We discuss the long-time behavior of the strong solutions so as to prove the existence of global compact attractor and exponential attractor.

1. Introduction and the main results

In this paper, we study the following initial–boundary value problem of the dissipative Boussinesq equation

$$u_{tt} - \Delta u + \Delta^2 u - \Delta u_t + \Delta^2 u_t = \Delta f(u) + g(x), \quad (x,t) \in \Omega \times \mathbb{R}^+ \tag{1.1}$$

$$u\,|_{\partial\Omega} = 0, \qquad \Delta u\,|_{\partial\Omega} = 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. $u(x,t)$ denotes the unknown function. Δ is the n-dimensional Laplace operator. The subscript t indicates the partial derivative with respect to t. u_0 and u_1 are the given initial value functions. $f \in C^2(\mathbb{R})$ is the given nonlinear function and $g(x)$ is an external force term.

Scott Russell [1] observed the existence of solitary waves, which are long, shallow, water waves of permanent form. This study motivated the development of nonlinear partial differential equations for the modeling wave phenomena in fluids, plasmas, elastic bodies, etc. Boussinesq [2] derived the first generalized

☆ This project is supported by the National Natural Science Foundation of China (Grant No. 11171311).

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http://dx.doi.org/10.1016/j.nonrwa.2016.03.002
1468-1218/

wave equation for the flow in shallow inviscid layer

$$u_{tt} - u_{xx} + \gamma u_{xxxx} = \beta(u^2)_{xx}, \tag{1.4}$$

where the constant coefficients γ and β depend on the depth of fluid and the characteristic speed of long waves. This was the first to give a scientific explanation of the existence to solitary waves found by Scott Russell.

In the case of $\gamma > 0$, Eq. (1.4) is known as the "good" Boussinesq (Bq) equation in comparison with the "bad" Bq equation defined as

$$u_{tt} = u_{xx} + c_1 u_{xxxx} + c_2(u^2)_{xx}, \quad c_1, c_2 > 0.$$

A generalization of one of the Bq type equations is considered which arises in the modeling of nonlinear strings, namely,

$$u_{tt} - u_{xx} + (u_{xx} + f(u))_{xx} = 0. \tag{1.5}$$

In [3], it has been proposed that certain, solitary wave solutions of (1.5) are nonlinearly stable for a range of their wave speeds.

For the "good" Bq equation and its generalized form, it is possible to study the well-posed problem, which is not the case for the "bad" version. In [3], Bona and Sachs obtained some sufficient conditions for the initial data to evolve into a global solution of Eq. (1.5). Linares [4] discussed the small-data Cauchy problem of Eq. (1.5). Linares and Scialom [5], considered the asymptotic behavior of the solutions for Eq. (1.5) with $f(u) = |u|^\alpha u$ $(\alpha > 1)$. The existence and nonexistence of global solutions of the generalized Bq equation (1.5) was investigated by Liu in [6]. Kishimoto [7] proved that the initial value problem of the "good" Bq equation is locally well-posed in $H^{-\frac{1}{2}}(\mathbb{T})$ and ill-posed in $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$. Cho and Ozawa [8] studied the existence and scattering of global small amplitude solutions of the multidimensional generalized Bq equation, and their results were extended by Ferreira [9].

Eq. (1.4) presents in appropriate balance between the nonlinearity and the dispersion because of the existence of solitary wave solutions (see [10]). Dissipation is naturally introduced in fluid dynamics through viscosity processes. In order to elucidate the interplay between the nonlinearity on one hand, and dispersion, energy input and dissipation on the other, the prototypical Bq-type of equation with dissipation and energy input then reads

$$u_{tt} = \left[\gamma^2 u - \frac{\alpha}{2}u^2 - \beta\gamma u_{xx} - \alpha_4 u_{xxt} - \alpha_2 u_t\right]_{xx} \tag{1.6}$$

which encompasses the oscillations of elastic beams. Here α is the amplitude coefficient, γ is the phase speed of the small disturbances, $(\beta\gamma)$ is the dispersion coefficient, α_2 is the coefficient of energy-production term, and α_4 is the dissipation coefficient. Rewritten (1.6) as a system

$$\begin{cases} v_t = q_{xx}, \\ q_t = \gamma^2 u - \dfrac{\alpha}{2}u^2 - \beta u_{xx} - \alpha_4 u_{xxt} - \alpha_2 u_t, \\ u = q_x = 0, \quad \text{for } x = -L_1, L_2 \end{cases}$$

where $-L_1, L_2$ are the values of the spatial coordinate at which truncate the infinite interval. The "mass" and energy of the wave system are as follows:

$$M \triangleq \int_{-L_1}^{L_2} u\mathrm{d}x,$$
$$E \triangleq \int_{-L_1}^{L_2} \frac{1}{2}\left(\gamma^2 u^2 + q_x^2 - \frac{1}{6}\alpha u^3 + \beta u_x^2\right)\mathrm{d}x,$$