Nonlinear Analysis: Real World Applications 31 (2016) 661–680

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

# Non-autonomous dynamics of a semi-Kolmogorov population model with periodic forcing

Tomás Caraballo[a,*], Renato Colucci[b], Xiaoying Han[c]

[a] *Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080 Sevilla, Spain*

[b] *Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, 111 Ren'ai Road, SIP, Suzhou 215123, PR China*

[c] *Department of Mathematics and Statistics, 221 Parker Hall, Auburn University, Auburn, AL 36849, USA*

ARTICLE INFO

*Article history:*
Received 8 July 2015
Accepted 15 March 2016

*Keywords:*
Nonautonomous dynamical system
Population dynamics
Pullback attractor

ABSTRACT

In this paper we study a semi-Kolmogorov type of population model, arising from a predator–prey system with indirect effects. In particular we are interested in investigating the population dynamics when the indirect effects are time dependent and periodic. We first prove the existence of a global pullback attractor. We then estimate the fractal dimension of the attractor, which is done for a subclass by using Leonov's theorem and constructing a proper Lyapunov function. To have more insights about the dynamical behavior of the system we also study the coexistence of the three species. Numerical examples are provided to illustrate all the theoretical results.



## 1. Introduction

Indirect effect refers to species interactions which can occur through chains of direct species interaction, such as predation or interference competition. The studies of indirect effects are of great importance to biological sciences, as they can link the population dynamics of species that do not interact directly (see [1–7] and references therein). The following system represents a typical population model (see [8–13]) that describes indirect effects of predation for one predator (zooplankton, denoted by $Z$) and two preys of different sizes (phytoplankton, denoted by $C$ and $G$):

$$\begin{cases} \dot{Z} = Z(-e + u_c C + u_g G), \\ \dot{C} = C[a_c I_0 - (a_c + u_c)Z - a_c C - a_c G] - m_1 CZ, \\ \dot{G} = G[a_g I_0 - (a_g + u_g)Z - a_g C - a_g G] + m_2 CZ. \end{cases} \tag{1.1}$$

* Corresponding author.
*E-mail addresses:* caraball@us.es (T. Caraballo), renatocolucci@hotmail.com (R. Colucci), xzh0003@auburn.edu (X. Han).

http://dx.doi.org/10.1016/j.nonrwa.2016.03.007
1468-1218/

The terms $-m_1CZ$ and $m_2CZ$ in system (1.1) describe the indirect effects generated by the fact that the predator prefers to predate the preys in the group with smaller size ($C$) and the other group ($G$) takes advantages of it. For the special case of $m_1 = m_2 = 0$, any positive solution leads to the extinction of at least one of the three species (see [8]). For $m_j > 0\,(j = 1, 2)$ uniform persistence has been shown, that there is a large variation of parameter sets with which any positive solution exhibits coexistence. In particular, sufficient conditions under which the attractor of the system is a fixed point or a stable limit cycle are found in [9] by using Hopf bifurcation.

Note that indirect effects can be of seasonal type (see for example [5,14]), thus it is natural to consider a non-autonomous counterpart of system (1.1) where constants $m_j$ are replaced by time dependent terms $m_j(t)$. A simple choice of the time dependent terms is:

$$m_j(t) := m_j|\sin(\omega t)|, \quad j = 1, 2. \tag{1.2}$$

The system is of particular interest when $m_2 > m_1$. In fact, summing the second and third equation of (1.1) with $m_j(t)$ as in (1.2) gives

$$\dot{C} + \dot{G} = (a_cC + a_GG)(I_0 - Z - C - G) - Z(u_cC + u_gG) + (m_2 - m_1)|\sin \omega t|, \tag{1.3}$$

for which we expect a more complicated behavior of the solutions as the last term of (1.3) is nonnegative.

Numerical simulations provided in [9] showed that with non-autonomous perturbation the system can exhibit not only the periodic orbit but also period bifurcation phenomena. It has also been observed (see e.g. [14] for a two dimensional example or [15] for a periodically forced system) that if the autonomous system has a periodic behavior then the non-autonomous system may behave similarly to forced oscillators [14].

In this paper we consider a generalized formulation of system (1.1):

$$\begin{cases} \dot{x}(t) = x(t)(-b_1 + a_{12}y(t) + a_{13}z(t)), \\ \dot{y}(t) = y(t)(b_2 - a_{21}x(t) - a_{22}y(t) - a_{23}z(t)) - g_1(t)x(t)y(t), \\ \dot{z}(t) = z(t)(b_3 - a_{31}x(t) - a_{32}y(t) - a_{33}z(t)) + g_2(t)x(t)y(t), \end{cases} \tag{1.4}$$

in which all the parameters $a_{i,j}, b_k\;(i, j, k = 1, 2, 3)$ are positive.

Here we assume that $g_1(t)$ and $g_2(t)$ in the periodic forcing terms satisfy

$$0 \le g_j(t) \le G_j, \quad \forall t \in \mathbb{R},\ j = 1, 2,$$

and $G_2 > G_1$. A possible choice of $g_j(t)$ arising from population models with indirect effects could be

$$g_j(t) = G_j|\sin(\omega_j t + \phi_j)|,$$

with $G_2 > G_1 > 0$ and $\omega_j, \phi_j > 0$, $(j = 1, 2)$. It is worth mentioning that it is possible to replace the $xy$ term in the forcing by a more general nonlinear form.

Observe that system (1.4) is not of the exact Kolmogorov type, as the third equation does not attain the Kolmogorov structure. Therefore this system can be regarded as a "semi-Kolmogorov-type" predator–prey system, which is different from traditional Kolmogorov systems, and is of more mathematical interests.

The rest of the paper is organized as follows. In Section 2 we provide some definitions and preliminary results from the theory of nonautonomous dynamical systems. In Section 3 we prove the existence of a pullback attractor for proper choice of the parameters. In Section 4 we estimate the Hausdorff dimension of the pullback attractor for a class of subsystems of (1.4). In Section 5 we study the coexistence of the three species, and a conclusion is given in Section 6.