Nonlinear Analysis: Real World Applications 17 (2014) 1–22

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

On the justification of the quasistationary approximation of several parabolic moving boundary problems—Part I

Friedrich Lippoth *

Institute of Applied Mathematics, Leibniz University of Hanover, Welfengarten 1, D-30167 Hannover, Germany

ARTICLE INFO

Article history:
Received 22 April 2013
Accepted 25 September 2013

ABSTRACT

We consider nonlinear coupled evolution equations evolving according to different timescales and study the behavior of solutions as their ratio becomes singular. We derive an abstract result and use it to justify rigorously the quasistationary approximation of a moving boundary problem modeling the growth of an avascular tumor. Another application is a quasilinear formulation of the Keller–Segel model on a bounded domain in $\mathbb{R}^N$.

1. Introduction

It is a well accepted strategy in the context of the analysis of dynamical systems, in particular moving boundary problems, to study so called quasistationary approximations rather than complete problems. This is reasonable, if the concrete problem couples the evolution of certain quantities with the property that one evolves significantly faster than the other. Moreover, the quasistationary problems are often easier to treat from a mathematical point of view. In this and a forthcoming paper we shall 'quantify' methods designed to solve the full parabolic systems in a way that is suitable to measure the difference of the involved timescales. As a consequence, we will be able not only to prove the local existence of solutions of (speed-) parameter dependent full problems on a uniform interval of existence, but also rigorously to prove strong convergence of the classical solutions to a solution of the corresponding quasistationary problem. We have decided to follow a systematic approach.

Section 2. We study parameter dependent abstract evolution systems $U_{\varepsilon,\alpha}(t,s)$ belonging to a family of time dependent operators of the form $\{1/\varepsilon \cdot A_\alpha(t);\, \varepsilon > 0;\, t \in [0,T];\, \alpha \in \mathsf{A}\}$ and derive sharp decay estimates explicitly respecting the parameter $\varepsilon > 0$ uniformly with respect to $\alpha \in \mathsf{A}$. Our systematic and notation is closely oriented on [1]. Our estimates allow us then to study the singular limit $\varepsilon = 0$ of the mild solution of the parameter dependent family of linear Cauchy problems

$$\varepsilon \dot{u} + A_\varepsilon(t) = F_\varepsilon(t); \qquad u(0) = u_0.$$

(Theorem 2.4) This is a generalization of an old result of S.G. Krein, see [2, Theorems IV.1.1 and IV.1.2], in miscellaneous respect.

Section 3. We consider an abstract coupled system of evolution equations

$$\begin{cases} \varepsilon \dot{u} + A(\rho)u = F(\varepsilon, u, \rho) \\ \dot{\rho} + B(\rho)\rho = G(u, \rho) \\ u(0) = u_0 \\ \rho(0) = \rho_0. \end{cases} \tag{1.1}$$

* Tel.: +49 5111231647.
E-mail address: lippoth@ifam.uni-hannover.de.

1468-1218/$ – see front matter
http://dx.doi.org/10.1016/j.nonrwa.2013.09.013

Under natural (from the point of view of semigroup theory) assumptions of the mappings involved we prove the local existence of classical solutions on an existence interval not depending on $\varepsilon > 0$ (Theorem 3.2). The latter thing turns out to be a crucial difficulty in the study of nonlinear coupled quantities evolving related to different timescales.

Section 4. We introduce a moving boundary problem modeling the growth of an avascular tumor and show that it fits in the framework designed before. The main result of this paper is to prove the strong convergence of classical solutions to the solution of the quasistationary approximation of the model (Theorem 4.5).

Section 5. As a byproduct of our abstract approach we show that a quasilinear version of the *Keller–Segel model* on a bounded domain in $\mathbb{R}^N$ admits smooth classical solutions rigorously approximating the quasistationary version.

We close this section with two remarks.

(1) There is an alternative strategy for attacking time-scaling problems of the type under our consideration. One can perform a scaling in the time variable and use suitable continuation operators in combination with maximal regularity results on the halfline (or decay estimates for evolution systems) in order to obtain a priori estimates. This strategy turns out to be technically even more involved, to be less consistent in its presentability and to produce less optimal results when dealing with the situation considered in this paper: First of all, the linear case (Theorem 2.4) would not be covered by this approach. Thus, in the situation of Theorem 3.2 we would not be able to establish E_1-convergence of the solutions which of course would force either to reason about the *notion* of a solution of the abstract quasistationary problem (3.1) or to work simultaneously in a second scale of Banach spaces representing higher regularity. Nevertheless, in a forthcoming paper we will make use also of these kinds of techniques in order to be able to justify the quasistationary approximations of a one phase osmosis model and of the Stefan problem with Gibbs–Thomson correction and kinetic undercooling.

(2) Recently I learned that V.A. Sollonikov and E.V. Frolova have rigorously justified the free boundary Stokes and Hele-Shaw models as singular limits of the Navier–Stokes system and the classical Stefan problem (i.e. the problem without Gibbs–Thomson correction and without kinetic undercooling), respectively [3,4]. Since they are using methods different from those presented in this paper (i.e. not based on semigroup theory), it is worth mentioning that (very roughly speaking) the estimates they derive in order to compare the two timescales are of similar nature as our ones, namely 'of the type $(\frac{t}{\varepsilon})^\alpha e^{-ct/\varepsilon}$'.

2. The abstract setting and linear equations

Let $\Sigma_\vartheta := \{z \in \mathbb{C};\, |\arg(z)| \le \vartheta + \pi/2\} \cup \{0\}$. If not explicitly otherwise stated, we assume throughout this section that

- E_1, E_0 are Banach spaces, $E_1 \overset{d}{\hookrightarrow} E_0$;
- J is a perfect subinterval of $\mathbb{R}^+$ containing 0, $0 < \rho < 1$;
- there are $M, \eta > 0$ as well as $\vartheta \in (0, \pi/2)$ such that

$$\begin{cases} \mathcal{A} \subset C^\rho(J, \mathcal{H}(E_1, E_0)), \qquad \|A\|_{C^\rho(J,\mathcal{L}(E_1,E_0))} \le \eta, \\ \Sigma_\vartheta \subset \rho(-A(s)) \\ \|A(s)\|_{\mathcal{L}(E_1,E_0)} + (1+|\lambda|)^{1-j}\|(\lambda + A(s))^{-1}\|_{\mathcal{L}(E_0,E_j)} \le M, \end{cases} \tag{2.1}$$

where $(s, \lambda, A) \in J \times \Sigma_\vartheta \times \mathcal{A}$ and $j = 0, 1$.

Here, $\mathcal{H}(E_1, E_0)$ denotes the set of all bounded linear operators $A \in \mathcal{L}(E_1, E_0)$, such that $-A$, considered as an unbounded operator in E_0 with domain E_1 generates a strongly continuous analytic semigroup of operators on E_0, i.e. in $\mathcal{L}(E_0)$, which we shall denote by e^{-tA}. The symbol $\rho(A)$ denotes the resolvent set of A, and, given metric spaces X, Y, $C^\rho(X, Y)$ is the set of ρ-Hölder continuous functions. (2.1) has the well-known implication

$$1/M\|x\|_{E_1} \le \|A(s)x\|_{E_0} \le M\|x\|_{E_1}, \tag{2.2}$$

$(s, x, A) \in J \times E_1 \times \mathcal{A}$. It is also well known that (2.1) guarantees the existence of a parabolic fundamental solution $U_A(t, s)$ possessing E_1 as a regularity subspace for any $A \in \mathcal{A}$. A detailed construction can be found in [1]. Moreover, estimates are proven, whose importance is revealed in the study of quasilinear problems. In the sequel we shall use the same techniques to derive estimates of the fundamental solutions belonging to a family of the form $\frac{1}{\varepsilon} \cdot A$, $\varepsilon > 0$, $A \in \mathcal{A}$. Indeed, Lemma III 2.2.1 in [1], (2.1), (2.2) and the fact that semigroup and generator commute, lead to the following statement.

Lemma 2.1. *There exist $C = C(k, M)$, $\sigma = \sigma(\vartheta, M) > 0$ such that*

$$\|[tA(s)]^k e^{-tA(s)}\|_{\mathcal{L}(E_j)} + t\|[tA(s)]^k e^{-tA(s)}\|_{\mathcal{L}(E_0,E_1)} \le C \cdot e^{-\sigma t}, \tag{2.3}$$

$k \in \mathbb{N}$, $(t, s, A) \in \mathbb{R}^{>0} \times J \times \mathcal{A}$, $j = 0, 1$.

Let X, Y, Z be Banach spaces and $J^*_\triangle := \{(t, s) \in J \times J;\, s < t\}$. If $f : J^*_\triangle \to \mathcal{L}(Y, Z)$ and $g : J^*_\triangle \to \mathcal{L}(X, Y)$ are suitable functions, let

$$(f \star g)(t, s) := \int_s^t f(t, \tau) g(\tau, s)\, d\tau$$