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Nonlinear stability for a simple model of a protoplanetary disc

B. Straughan

Department of Mathematical Sciences, Durham University, DH1 3LE, UK

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ABSTRACT

We study a simple one space dimensional model which has been proposed to describe the behaviour of a protoplanetary disc. The basic model arises from the induction equation for the magnetic field in such a disc. Firstly we analyse the stability and instability thresholds when a shear is imposed and the Hall effect is present. In this situation we employ a classical variational energy method and a transformation variant which achieves an optimal bound. Secondly we also include the effect of ion-slip. A variational energy technique is shown to yield a sharp nonlinear stability threshold.

1. Introduction

Rüdiger and Shalybkov [1] have proposed a simple one-dimensional model for instability in a protoplanetary disc. Their model begins with the induction equation for the magnetic induction field **B**, namely,

$$\frac{\partial B_i}{\partial t} = \left[\text{curl}\,(\mathbf{u} \times \mathbf{B})\right]_i + \eta \Delta B_i - \beta \left[\text{curl}(\text{curl}\,\mathbf{B} \times \mathbf{B})\right]_i. \tag{1}$$

Here $\mathbf{u}$ is an imposed velocity field, η is the magnetic diffusivity, β is a Hall coefficient with the last (nonlinear) term in Eq. (1) representing the Hall effect, see e.g. Cowling [2], Fabrizio and Morro [3, p. 292]. The influence of the Hall effect on the stability of a protoplanetary disc and similar geometries is a subject of much recent interest, and further details may be found in the papers Desch [4], Jia and Zhou [5], Liverts et al. [6], Prudskikh [7], Rüdiger and Kitchatinov [8], Sano and Stone [9], and Vajravelu et al. [10]. Rüdiger and Shalybkov [1] assume that the velocity field is given of form $\mathbf{u} = (0, u_0 x, 0)$ for u_0 a constant, so that the magnetic field is subject to a linear velocity shear effect. The magnetic field studied has form $\mathbf{B} = (B_x(z, t), B_y(z, t), B_0)$ where B_0 is a constant. If one scales the magnetic induction field with B_0, i.e. $B_x = B_0 u,\ B_y = B_0 v$, then with a disc thickness $2H$ in the z-direction, one shows, cf. Rüdiger and Shalybkov [1] Eq. (3), that the functions $u(z, t), v(z, t)$ satisfy the non-dimensional equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial z^2} + C_H \frac{\partial^2 v}{\partial z^2}, \\ \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial z^2} + C_\Omega u - C_H \frac{\partial^2 u}{\partial z^2}, \end{aligned} \tag{2}$$

where the non-dimensional parameters C_H and C_Ω are given by

$$C_H = \frac{\beta B_0}{\eta} \qquad C_\Omega = \frac{H^2 u_0}{\eta},$$

with time scaled as H^2/η.

E-mail address: brian.straughan@durham.ac.uk.

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The boundary conditions for u and v are

$$u(\pm 1) = 0, \qquad v(\pm 1) = 0. \tag{3}$$

Rüdiger and Shalybkov [1] analyse the instability of the zero solution to (2), (3) and discover that the transition to instability threshold is given by

$$-C_H C_\Omega > \frac{\pi^2}{4} C_H^2 + \frac{\pi^2}{4}. \tag{4}$$

Our goal here is to show that solutions to (2), (3) decay exponentially fast if condition (4) is negated provided some other weak bounds are imposed. We achieve this by employing energy methods. One of the goals of this paper is thus to find an optimal Lyapunov functional (or energy functional) in the theory of nonlinear stability in partial differential equations. This sort of analysis originated with Joseph's [11] early work on the double diffusive convection problem. The point is that a standard L^2 type "energy functional" fails to yield an optimal nonlinear stability result for many fluid mechanics problems such as the heated and salted below double diffusive convection problem studied by Joseph [11]. Producing an alternative functional often requires ingenuity and this has resulted in intense effort including increased recent attention, cf. Capone and de Luca [12], Falsaperla et al. [13], Galdi and Straughan [14,15], Galdi et al. [16], Georgescu and Palese [17], Hill and Malashetty [18], Lombardo et al. [19], Mulone et al. [20], Payne and Straughan [21,22], Pieters and van Duijn [23], Rionero [24–26], Straughan [27], van Duijn et al. [28]. We also consider a more general, highly nonlinear adaptation of (1) or (2), to include the ion-slip effect, in Section 4.

It is convenient to first rescale (2), (3) from $\hat{z} \in (-1, 1)$ to $z \in (0, 1)$. This means we study the boundary initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{4}\frac{\partial^2 u}{\partial z^2} + \frac{C_H}{4}\frac{\partial^2 v}{\partial z^2}, \\ \frac{\partial v}{\partial t} &= \frac{1}{4}\frac{\partial^2 v}{\partial z^2} + C_\Omega u - \frac{C_H}{4}\frac{\partial^2 u}{\partial z^2}, \end{aligned} \tag{5}$$

where $t > 0, z \in (0, 1)$, and

$$u = 0, \qquad v = 0, \qquad z = 0, 1, \tag{6}$$

with $u(z, 0), v(z, 0)$ prescribed.

2. Energy stability in a classical norm

Let us observe that since (5) is linear and (6) hold we may find the instability threshold by looking for solutions of the form $u = \hat{u}\, e^{\sigma_n t} \sin n\pi z$, $v = \hat{v}\, e^{\sigma_n t} \sin n\pi z$, (actually a Fourier series, or more accurately a Fourier transform, of such terms). This quickly leads to a dominant eigenvalue relation of form

$$\sigma_1 = -\frac{\pi^2}{4} \pm \sqrt{-\frac{\pi^2}{4} C_H\left(C_\Omega + \frac{C_H}{4}\pi^2\right)}. \tag{7}$$

Clearly one of these leads to decay and instability is found with the plus sign when the Rüdiger and Shalybkov [1] criterion (4) holds. Rüdiger and Shalybkov [1] also show that (4) leads to

$$|C_\Omega| > \frac{\pi^2}{4}\left(\frac{1 + C_H^2}{|C_H|}\right) \tag{8}$$

which has an absolute minimum at

$$|C_\Omega| = \frac{\pi^2}{2} \quad \text{when } |C_H| = 1. \tag{9}$$

We first establish a decay result when $|C_\Omega|$ is less than the bound in (9). Let $\|\cdot\|$ and $(\cdot, \cdot)$ denote the norm and inner product on $L^2(0, 1)$.

Theorem 1. *If the constant C_Ω satisfies*

$$|C_\Omega| < \frac{\pi^2}{2} \tag{10}$$

then the solutions u, v to (5) and (6) decay at least exponentially fast in time in $L^2(0, 1)$ norm.