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Global solutions to phase change models with heat transfer for a class of smart materials

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ABSTRACT

This paper is concerned with the global solvability of three-dimensional models for smart materials undergoing phase transitions with heat transfer. The problem is formulated within the framework of generalized standard solids by the coupling of the momentum equilibrium equation and the flow rule with the heat transfer equation. Under appropriate regularity assumptions on the initial data, a global existence result for this thermodynamically consistent system is established by using a fixed-point argument combined with energy estimates.

1. Introduction

Shape-memory materials belong to the general class of so-called smart materials. These materials severely deformed have the ability to recover their original shape after a thermal cycle (shape-memory effect). Their use in innovative and commercially valuable applications has motivated several studies on their physical properties during the last decade (see for instance [1–3] and the references therein). We consider in this work some three-dimensional models for smart materials undergoing phase transitions. In the framework of generalized standard solids due to Halphen and Nguyen (see [4]), the unknowns are the displacement field u, an internal variable z and the temperature θ and the problem is described by the coupling of the heat equation with the momentum equilibrium equation combined with a constitutive law (flow rule) for the evolution of the internal variable.

More precisely let us denote by $W(\mathrm{e}(u), z, \nabla z, \theta)$ the Helmholtz free energy, depending on the infinitesimal strain tensor $\mathrm{e}(u) \stackrel{\text{def}}{=} \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$ for the displacement u, where $(\cdot)^{\mathsf{T}}$ denotes the transpose of the tensor, the internal variable z and the temperature θ [5,4]. For simplicity, we will omit any dependence on the material point $x \in \Omega$ and $t \in [0, T]$ with $T > 0$. We assume that W can be decomposed as follows

$$W(\mathrm{e}(u), z, \nabla z, \theta) \stackrel{\text{def}}{=} W_1(\mathrm{e}(u), z, \nabla z) - W_0(\theta) + \theta W_2(\mathrm{e}(u), z). \tag{1.1}$$

This partially linearized decomposition ensures that the entropy separates the thermal and mechanical variables (see (2.8)). Moreover the last term $\theta W_2(\mathrm{e}(u), z)$ allows for coupling effects between the temperature and the internal variable, which is motivated by some phenomenological models for shape memory alloys presented in Section 2. We make the assumptions of small deformations. The problem is thus described by the following system

$$-\operatorname{div}(\sigma_{\mathrm{el}} + \mathbb{L}\mathrm{e}(\dot{u})) = \ell, \quad \sigma_{\mathrm{el}} \stackrel{\text{def}}{=} \mathrm{D}_{\mathrm{e}(u)} W(\mathrm{e}(u), z, \nabla z, \theta), \tag{1.2a}$$

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$$\partial\Psi(\dot z) + \mathbb{M}\dot z + \sigma_{\rm in} \ni 0,\ \sigma_{\rm in} \stackrel{\rm def}{=} \mathrm{D}_z W(\mathrm{e}(u), z, \nabla z, \theta) - \operatorname{div} \mathrm{D}_{\nabla z} W(\mathrm{e}(u), z, \nabla z, \theta), \tag{1.2b}$$

$$c(\theta)\dot\theta - \operatorname{div}(\kappa(\mathrm{e}(u), z, \theta)\nabla\theta) = \mathbb{L}\mathrm{e}(\dot u) : \mathrm{e}(\dot u) + \theta\partial_t W_2(\mathrm{e}(u), z) + \Psi(\dot z) + \mathbb{M}\dot z : \dot z. \tag{1.2c}$$

Here Ψ is a dissipation potential. As it is common in modeling hysteresis effect in mechanics, we assume that Ψ is positively homogeneous of degree 1, i.e., $\Psi(\gamma z) = \gamma\Psi(z)$ for all $\gamma \geq 0$. The viscosity tensors are denoted by $\mathbb{L}$ and $\mathbb{M}$, $c(\theta)$ is the heat capacity and $\kappa(\mathrm{e}(u), z, \theta)$ is the conductivity. As usual, $\dot{(\cdot)}$, D_z^i and ∂ denote the time derivative $\frac{\partial}{\partial t}$, the i-th derivative with respect to z and the subdifferential in the sense of convex analysis [6,7], respectively. Observe that (1.2a)–(1.2c) are usually called the momentum equilibrium equation, the flow rule and the heat-transfer equation, respectively.

The paper is organized as follows. In Section 2, we justify the thermodynamic consistency of this model and we describe it more completely. Then the mathematical formulation of the problem in terms of displacement, internal variables and temperature is introduced in Section 3, as well as a reformulation of the heat equation in terms of enthalpy. Sections 4–6 are devoted to the proof of a local existence result by a fixed point argument. More precisely, in Section 4 existence and uniqueness results for the system composed of the momentum equation and the flow rule for a given temperature are recalled and some regularity results for the solutions (u, z) are established. Section 5 is dedicated to recall existence and regularity results for the enthalpy equation for any given right hand side. Therefore a local existence result follows in Section 6 by using a fixed-point argument. Finally a global energy estimate is established in Section 7 leading to a global existence result for the system (1.2).

2. Mechanical model

We justify here the thermodynamic consistency of the model (1.2). Starting from the Helmholtz free energy W, we introduce the specific entropy s via Gibb's relation

$$s \stackrel{\rm def}{=} -\mathrm{D}_\theta W(\mathrm{e}(u), z, \nabla z, \theta), \tag{2.1}$$

and the internal energy

$$W_{\rm in}(\mathrm{e}(u), z, \nabla z, \theta) \stackrel{\rm def}{=} W(\mathrm{e}(u), z, \nabla z, \theta) + \theta s. \tag{2.2}$$

Then the entropy equation is given by

$$\theta\dot s + \operatorname{div}(j) = \xi, \tag{2.3}$$

where j is the heat flux and ξ is the dissipation rate. We get

$$\xi = \mathbb{L}\mathrm{e}(\dot u) : \mathrm{e}(\dot u) + \mathbb{M}\dot z : \dot z + \Psi(\dot z) \geq 0,$$

and, assuming Fourier's law for the temperature, we have

$$j = -\kappa(\mathrm{e}(u), z, \theta)\nabla\theta.$$

We can check now that the second law of thermodynamics is satisfied if $\theta > 0$. Indeed, assuming that the system is thermally isolated, we may divide (2.3) by θ and Green's formula yields

$$\begin{aligned}\int_\Omega \dot s\,\mathrm{d}x &= \int_\Omega \frac{\operatorname{div}(\kappa(\mathrm{e}(u), z, \theta)\nabla\theta)}{\theta}\,\mathrm{d}x + \int_\Omega \frac{\mathbb{L}\mathrm{e}(\dot u) : \mathrm{e}(\dot u) + \mathbb{M}\dot z : \dot z + \Psi(\dot z)}{\theta}\,\mathrm{d}x \\ &= \int_\Omega \frac{\kappa(\mathrm{e}(u), z, \theta)\nabla\theta\cdot\nabla\theta}{\theta^2}\,\mathrm{d}x + \int_\Omega \frac{\mathbb{L}\mathrm{e}(\dot u) : \mathrm{e}(\dot u) + \mathbb{M}\dot z : \dot z + \Psi(\dot z)}{\theta}\,\mathrm{d}x \geq 0.\end{aligned}$$

We differentiate now $W_{\rm in}(\mathrm{e}(u), z, \nabla z, \theta)$ with respect to time, and we obtain by using the chain rule and (2.1) that

$$\begin{aligned}\dot W_{\rm in}(\mathrm{e}(u), z, \nabla z, \theta) = {}& \mathrm{D}_{\mathrm{e}(u)} W(\mathrm{e}(u), z, \nabla z, \theta) : \mathrm{e}(\dot u) + \mathrm{D}_z W(\mathrm{e}(u), z, \nabla z, \theta) : \dot z \\ &+ \mathrm{D}_{\nabla z} W(\mathrm{e}(u), z, \nabla z, \theta)\cdot\nabla\dot z + \theta\dot s.\end{aligned} \tag{2.4}$$

We integrate (2.4) over Ω, thus we use Green's formula and (2.3), and we find

$$\begin{aligned}\int_\Omega \dot W_{\rm in}(\mathrm{e}(u), z, \nabla z, \theta)\,\mathrm{d}x = {}& \int_\Omega \mathrm{D}_{\mathrm{e}(u)} W(\mathrm{e}(u), z, \nabla z, \theta) : \mathrm{e}(\dot u)\,\mathrm{d}x \\ &+ \int_\Omega \mathrm{D}_z W(\mathrm{e}(u), z, \nabla z, \theta) : \dot z\,\mathrm{d}x + \int_\Omega \mathrm{D}_{\nabla z} W(\mathrm{e}(u), z, \nabla z, \theta)\cdot\nabla\dot z\,\mathrm{d}x \\ &+ \int_\Omega (\operatorname{div}(\kappa(\mathrm{e}(u), z, \theta)\nabla\theta) + \mathbb{L}\mathrm{e}(\dot u) : \mathrm{e}(\dot u) + \mathbb{M}\dot z : \dot z + \Psi(\dot z))\,\mathrm{d}x.\end{aligned} \tag{2.5}$$

On the one hand, we multiply (1.2a) by $\dot u$, and we integrate this expression over Ω to get

$$\int_\Omega \mathrm{D}_{\mathrm{e}(u)} W(\mathrm{e}(u), z, \nabla z, \theta) : \mathrm{e}(\dot u)\,\mathrm{d}x + \int_\Omega \mathbb{L}\mathrm{e}(\dot u) : \mathrm{e}(\dot u)\,\mathrm{d}x = \int_\Omega \ell\cdot\dot u\,\mathrm{d}x. \tag{2.6}$$