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On the symmetry and periodicity of solutions of differential systems[☆]

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ABSTRACT

This article deals with the structure of the reflective function of the higher degree polynomial differential systems. The obtained results are applied to discussion of the symmetry and periodicity of the solutions of these systems.

1. Introduction

Studying the property of the solutions of the differential system

$$x' = X(t, x) \tag{1}$$

is very important not only for the theory of ordinary differential equations but also for practical reasons. If $X(t + 2\omega, x) = X(t, x)$ (ω is a positive constant), to study the solutions behavior of (1), we could use, as introduced in [1], the Poincaré mapping. But it is very difficult to find the Poincaré mapping for many systems which cannot be integrated in quadratures. The Russian mathematician Mironenko [1,2] first established the theory of reflective functions. Since then a new method to establish the Poincaré mapping of (1) has been found.

In the present section, we introduce the concept of the reflective function, which will be used throughout the rest of this article.

Consider differential system (1) which has a continuous differentiable right-hand side and general solution $\varphi(t; t_0, x_0)$. For each such system, the reflecting function (**RF**) is defined as $F(t, x) := \varphi(-t, t, x)$ [1,2]. Therefore, for any solution $x(t)$ of (1), we have $F(t, x(t)) = x(-t)$.

If system (1) is 2ω-periodic with respect to t, then $T(x) := F(-\omega, x) = \varphi(\omega; -\omega, x)$ is the Poincaré mapping of (1) over the period $[-\omega, \omega]$. Thus, the solution $x = \varphi(t; -\omega, x_0)$ of (1) defined on $[-\omega, \omega]$ is 2ω-periodic if and only if x_0 is a fixed point of $T(x)$. The stability of this periodic solution is equivalent to the stability of the fixed point x_0.

A differentiable function $F(t, x)$ is a reflecting function of system (1) if and only if it is a solution of the Cauchy problem

$$F'_t + F'_x X(t, x) + X(-t, F) = 0, \qquad F(0, x) = x. \tag{2}$$

This implies that sometimes for non-integrable periodic systems we can find out its Poincaré mapping. If, for example, $X(t, x) + X(-t, x) = 0$, then $T(x) = x$.

If $F(t, x)$ is the **RF** of (1), then it is also the **RF** of the system

$$x' = X(t, x) + F_x^{-1} R(t, x) - R(-t, F(t, x))$$

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where $R(t, x)$ is an arbitrary vector function such that the solutions of the above systems are uniquely determined by their initial conditions. Therefore, all these 2ω-periodic systems have a common Poincaré mapping over the period $[-\omega, \omega]$, and the behavior of the periodic solutions of these systems are the same. So, to find out the reflective function is very important for studying the qualitative behavior of solutions of differential systems.

There are many papers which are also devoted to investigations of qualitative behavior of solutions of differential systems by help of reflective functions. Mironenko [1–5] combined the theory of **RF** with the integral manifolds theory to discuss the symmetry and other geometric properties of solutions of (1), and obtained a lot of excellent new conclusions. Alisevich [6] has discussed when a linear system has triangular reflective function. Musafirov [7] has studied when a linear system has reflective function which can be expressed as a product of three exponential matrices. Veresovich [8] has researched when the nonautonomous two-dimensional quadric systems are equivalent to a linear system. Maiorovskaya [9] has established the sufficient conditions under which the quadratic systems have linear reflecting function. Zhou [10,11] has discussed the structure of reflective function of quadratic systems, and applied the obtained conclusions to study the qualitative behavior of solutions of differential systems.

Now, we consider a higher degree polynomial differential system

$$\begin{cases} \dfrac{dx}{dt} = \displaystyle\sum_{i=0}^{n} p_i(t, x)y^i = P(t, x, y), \\ \dfrac{dy}{dt} = \displaystyle\sum_{j=0}^{m} q_j(t, x)y^j = Q(t, x, y), \end{cases} \tag{3}$$

where $p_i(t, x); q_j(t, x)$ $(i = 0, 1, 2, \ldots, n,\ j = 0, 1, 2, \ldots, m)$ are continuously differentiable functions in R^2 and $\frac{\partial P(t,x,y)}{\partial y} \neq 0$, and there exists a unique solution for the initial value problem of (3). n and m are positive integers.

In the following, suppose that $F(t, x, y) = F_1(t, x, y), F_2(t, x, y)^T$ is the **RF** of (3). In this paper, we will discuss the structure of $F_2(t, x, y)$ when $F_1(t, x, y) = f(t, x)$. At the same time, we obtain the good result that $F_2(t, x, y) = f_0(t, x) + f_1(t, x)y$, which extends and improves the conclusion of literatures [10,11]. The obtained results are used for research of problems of the existence of a periodic solution of system (3) and establish the sufficient conditions under which the first component of the solution of (3) is an even function with respect to time t. The obtained conclusions extend and improve the known results.

2. Main results

Without loss of generality, we assume that $f(t, x) = x$. Otherwise, we take the transformation $u = f(t, x)$, $v = y$. Let $F_1(t, x, y) = x$. Then by the relation (2) we get $P(t, x, y) + P(-t, x, F_2(t, x, y)) = 0$, i.e.,

$$\sum_{i=0}^{n} A_i F_2^i = 0, \tag{4}$$

where

$$A_0 = \frac{1}{p_n(-t, x)}[p_0(-t, x) + P(t, x, y)]; \qquad A_i = \frac{p_i(-t, x)}{p_n(-t, x)}, \quad i = 1, 2, \ldots, n-1.$$

Here suppose that in some deleted neighborhood of $t = 0$ and $|t|$ being small enough $p_n(t, x)$ is different from zero.

As $F_2(0, x, y) = y$, from (4) implies the following lemmas directly.

Lemma 1. *For the system (3). If $F_1 = x$, then*

$$p_i(0, x) = 0, \quad i = 0, 1, 2, \ldots, n. \tag{5}$$

Lemma 2. *Let for the system (3) $F_1 = x$ and $\lim_{t\to 0} \frac{p_i(t,x)}{p_n(t,x)}$ $(i = 1, 2, \ldots, n-1)$ exist. Then*

$$\lim_{t\to 0} \frac{p_i(t, x) + p_i(-t, x)}{p_n(t, x)} = 0 \quad (i = 0, 1, 2, \ldots, n-1), \qquad \lim_{t\to 0} \frac{p_n(t, x)}{p_n(-t, x)} = -1. \tag{6}$$

In the following discussion, we always assume (5) holds without further mention.

Theorem 1. *Suppose that $F = (x, F_2)^T$ is the **RF** of system (3) and F_2 satisfies*

$$F_2^2 + A_1 F_2 + A_0 = 0, \tag{7}$$