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Stabilization of free surface flows in periodic porous media

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ABSTRACT

The flow of an incompressible Newtonian fluid in a periodic porous medium is studied. The equations governing this motion form a free boundary problem involving a potential flow problem and the evolution of a free surface. The flow is driven by gravitational forces and it is stabilized by an external source. The existence of a unique classical solution for large initial data is proved in the sense that there is an unbounded set of initial conditions for which a unique classical solution exists.

1. Introduction and main results

We study the laminar flow of an incompressible Newtonian fluid in a homogeneous isotropic porous medium under the influence of gravity. We consider a periodic geometry, i.e., the free surface of the fluid is assumed to be the graph of a function over the n-torus $\mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$ and we identify these functions with 2π-periodic functions. Given $\tilde{f} \in C^2(\mathbb{T}^n)$, we define the free surface as

$$\Gamma(\tilde{f}) := \{(x, \tilde{f}(x)) : x \in \mathbb{T}^n\}.$$

The fluid covers the domain

$$\Omega(\tilde{f}) := \{(x, y) \in \mathbb{T}^n \times \mathbb{R} : 0 < y < \tilde{f}(x)\}$$

having the fixed bottom part $\Gamma_0 := \mathbb{T}^n \times \{0\} \equiv \mathbb{T}^n$. For the flow we assume a Darcy law to hold, i.e., the velocity of the fluid is given by $-K\nabla_{n+1}u$, where K is a material constant and u is a velocity potential. If we assume the fluid to be incompressible we get $-\mathrm{div}_{n+1}K\nabla_{n+1}u = 0$, i.e., $\Delta_{n+1}u = 0$. We normalize the air pressure at the free surface to be zero and are then able to write down the velocity potential at the free surface as $u = \tilde{f}$. At Γ_0 we model an external source $\tilde{b}$ being a real analytic function from $\mathbb{R}$ to $\mathbb{R}$; see [1,2] for similar models. Adding a transport equation for the free surface with initial condition $\tilde{f}_0$, we obtain the free boundary problem

$$\begin{array}{ll} \Delta_{n+1}u = 0 & \text{in } \Omega(\tilde{f}), \\ (1-\delta)u + \delta\partial_{n+1}u = \tilde{b}(\tilde{f}) & \text{on } \Gamma_0, \\ u = \tilde{f} & \text{on } \Gamma(\tilde{f}), \\ \partial_t\tilde{f} = -\sqrt{1+|\nabla_n f|^2}\,\partial_\nu u & \text{on } \Gamma(\tilde{f}), \\ \tilde{f}(0) = \tilde{f}_0 & \text{on } \mathbb{T}^n, \end{array} \tag{1}$$

where $\nu := N/\|N\|$ is the unit outward normal at $\Gamma(\tilde{f})$ with $N := (-\nabla_n f, 1)$. The parameter $\delta \in \{0, 1\}$ is fixed from the very beginning. It determines whether a Dirichlet ($\delta = 0$) or a Neumann ($\delta = 1$) boundary condition is considered. In case

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$\delta = 0$ the pressure is prescribed at the lower boundary component whereas in case $\delta = 1$ a rate of injection/suction is given by $\tilde{b}$. Note that the above model can also be considered as a one-phase Hele-Shaw model in a strip-like geometry.

Free boundary problems of this (or similar) kind have attracted much attention during recent years. Dating back to the works of Kawarada and Koshigoe [3] as well as Escher and Simonett [4] the abstract theory of these free boundary problems has continuously become more diverse. Whereas in the articles mentioned above the fluid is—like in our model—of Newtonian type, also non-Newtonian fluids have been studied in more recent years, see, e.g., [5,2]. As in our work a periodic geometry is considered there, cf. also [6]. In contrast, in [3,4] the geometry is unbounded.

Appropriate techniques to solve these free boundary problems are methods from the theory of analytic semigroups and abstract parabolic equations. The original problem is reduced to an abstract Cauchy problem in appropriate function spaces for the free surface only. Then the linearization of the corresponding evolution operator is studied. If it can be shown that, given some initial condition, it generates an analytic semigroup on the respective function space the short time existence of a unique classical solution can be deduced from maximal regularity arguments.

The existing well-posedness results for periodic free surface flows of the above (or similar) type are usually local results in the sense that it is not known how "large" the set of initial conditions is for which the corresponding Cauchy problem is parabolic. This is the case, e.g., if the existence of classical solutions is studied only near some trivial solution for which it is quite accessible to establish such a generation result. Then perturbation arguments can be applied to extend this generation result to a neighborhood of that trivial solution. We aim to improve these existing local generation results by explicitly describing an in some way large set of initial conditions such that the corresponding evolution operator generates an analytic semigroup.

To begin with we introduce the basic function spaces in which we formulate our results. Given $s > 0$, the little Hölder space over the n-torus $h^s(\mathbb{T}^n)$ is defined as the closure of the smooth functions $C^\infty(\mathbb{T}^n)$ in the space $C^s(\mathbb{T}^n)$ of (Hölder) continuous functions with norm $\|\cdot\|_s$. An equivalent intrinsic definition of these spaces shows that they are true subspaces of the spaces $C^s(\mathbb{T}^n)$. Moreover, these spaces are densely injected, i.e., $h^t(\mathbb{T}^n) \overset{d}{\hookrightarrow} h^s(\mathbb{T}^n)$ for $t > s$. Since this relation does not hold for the spaces $C^s(\mathbb{T}^n)$, $s > 0$, we work in little Hölder spaces.

Let $\mathrm{Ad}_y := \{f \in C^2(\mathbb{T}^n) : \min_{x\in\mathbb{T}^n} f(x) > y\}$ and, given $s > 0$, define $h^s_+(\mathbb{T}^n) := h^s(\mathbb{T}^n) \cap \mathrm{Ad}_0$. Note that $h^s_+(\mathbb{T}^n)$ is the cone of strictly positive elements in $h^s(\mathbb{T}^n)$. Furthermore, let us drop the n-torus from our notation and define $h^s := h^s(\mathbb{T}^n)$ and $h^s_+ := h^s_+(\mathbb{T}^n)$ for $s > 0$. Finally, for $s > 0$, $m \in \mathbb{N}$, and an open and bounded subset $X \subset \mathbb{R}^m$ with a C^2 boundary the space $buc^s(X)$ is defined as the closure of the bounded and uniformly continuous functions of arbitrary order $BUC^\infty(X)$ in the space of bounded and uniformly (Hölder) continuous functions of degree s on X denoted by $BUC^s(X)$ with norm $\|\cdot\|_{s,X}$. Observe the analogy of this definition to the definition of the spaces h^s, $s > 0$.

We call a pair $(\tilde{f}, u)$ a classical Hölder solution of problem (1) if there is $T > 0$ such that

$$\tilde{f} \in C([0,T), h^{2+\alpha}_+) \cap C^1([0,T), h^{1+\alpha}),$$
$$u(\cdot,t) \in buc^{2+\alpha}(\Omega(\tilde{f}(t,\cdot))), \quad 0 \le t \le T,$$

and $\tilde{f}$, u fulfill (1) pointwise. Given $\tilde{f} \in h^{2+\alpha}_+$, put

$$\kappa_{\tilde{f}} := \frac{\tilde{f}^2}{(1+\tilde{f}^2+|\nabla_n\tilde{f}|^2)(1+|\nabla_n\tilde{f}|^2)}$$

and let $u_{\tilde{f}}$ be the solution of

$$\begin{aligned} &\Delta_{n+1}u = 0 && \text{in } \Omega(\tilde{f}),\\ &(1-\delta)u + \delta\partial_{n+1}u = \tilde{b}(\tilde{f}) && \text{on } \Gamma_0,\\ &u = \tilde{f} && \text{on } \Gamma(\tilde{f}). \end{aligned} \tag{2}$$

Finally, defining the set

$$\mathcal{W} := \{\tilde{f} \in h^{2+\alpha}_+ : \partial_{n+1}u_{\tilde{f}}(x,\tilde{f}(x)) < \kappa_{\tilde{f}}(x), x \in \mathbb{T}^n\}, \tag{3}$$

we can formulate our main result.

Theorem 1. *Given $c > 0$, suppose that $(-1)^\delta\tilde{b}(c) > (-1)^\delta c^{1+\delta}/(1+c^2)$. For every $\tilde{f}_0 \in \mathcal{W}$ there exists $T > 0$ and a unique maximal classical Hölder solution $(\tilde{f}, u)$ of system (1) on $[0,T)$.*

2. The transformation

In this section we transform system (1) on a fixed reference domain where we perform its further analysis. This transformation introduces nonlinear terms in the differential operators in the following way. Let $\alpha \in (0,1)$ and $c > 0$ be fixed in the following and let $\tilde{f} \in h^{2+\alpha}_+$ be given. Put

$$\tilde{\phi}_{\tilde{f}}(x,y) := (x, 1 - y/\tilde{f}(x)), \quad (x,y) \in \Omega(\tilde{f}).$$