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Morse theory and local linking for a nonlinear degenerate problem arising in the theory of electrorheological fluids

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ABSTRACT

Many electrorheological fluids are suspensions of solid particles that are exposed to a strong electric field. This causes a dramatic increase of their effective viscosity. In this paper we are concerned with a mathematical problem that is related with this non-Newtonian behavior. More precisely, we study the nonlinear stationary equation $-\text{div}\,(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = f(x, u)$ in Ω, under Dirichlet boundary conditions, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $p > 1$ is a continuous function, and $f(x, u)$ has a sublinear growth near the origin. Under various natural assumptions, by using the Morse theory in combination with local linking arguments, we obtain the existence of nontrivial weak solutions.

1. Introduction and main results

The qualitative analysis of nonlinear partial differential equations involving differential operators with variable exponent is motivated by wide applications to various fields. Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery on electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their effective viscosity depends on the electric field in the fluid. The dramatic increase of the effective viscosity (or shear stress) is due to the existence of special particle structures that appear in the presence of an electric field hindering the flow. Winslow noticed that in such fluids (for instance lithium polymethacrylate) the effective viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the effective viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect*. For a general account of the underlying physics we refer to Halsey [1]. For overviews of microscopic models in relationship with applications to electrorheology we refer the reader to Parthasarathy and Klingenberg [2] and Růžička [3]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. We also point out an interesting recent mathematical model developed by Rajagopal and Růžička [4]. The model takes into account the delicate interaction between the electromagnetic fields and the moving fluids. Particularly, in the context of continuum mechanics, these fluids are seen as non-Newtonian fluids. Other relevant applications of nonlinear equations involving differential operators with variable exponent include obstacle problems [5,6], porous medium equation [7–10], and Kirchhoff problems [11,12].

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In the recent paper [13] it is introduced a method to study the existence and multiplicity of solutions for a class of nonlinear stationary partial differential equations described by a non-standard differential operator with variable exponent. We continue this analysis in the present paper, where we describe how powerful tools in modern nonlinear analysis (Morse theory and linking theory) can be used to establish some qualitative properties of solutions to these equations.

1.1. Statement of the problem

The goal of this paper is to investigate the existence of nontrivial solutions to the following $p(x)$-Laplacian problem:

$$\begin{cases} -\triangle_{p(x)} u + |u|^{p(x)-2}u := -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $p \in C(\overline{\Omega})$ and $1 < p_- := \min_{x\in\bar{\Omega}} p(x) \le p(x) \le p_+ := \max_{x\in\bar{\Omega}} p(x) < \infty$ and $F(x,t) = \int_0^t f(x,s)ds$, $\mathcal{F}(x,t) = f(x,t)t - p_+F(x,t)$. For simplicity, we denote by C and C_k various positive constants whose exact value is irrelevant.

We assume that the reaction term $f(x,u)$ satisfies the following hypotheses:

(H1) $f \in C(\overline{\Omega} \times \mathbb{R})$ with $f(x,0) = 0$ and there exists $C_1 > 0$ such that

$$|f(x,t)| \le C_1(1 + |t|^{q(x)-1}), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $q(x) \in C(\overline{\Omega})$, $1 < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $p^* = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$, $p^*(x) = +\infty$ if $p(x) \ge N$;

(H2) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{p_+}} = +\infty$ uniformly for $x \in \bar{\Omega}$;

(H3) there exists $\theta \ge 1$ such that $\theta\mathcal{F}(x,t) \ge \mathcal{F}(x,st)$ for $(x,t) \in \Omega \times \mathbb{R}$ and $s \in [0,1]$;

(H4) there exists $\nu > 0$ such that

$$\frac{f(x,t)}{|t|^{p_+-2}t} \text{ is increasing in } t \ge \nu \text{ and decreasing in } t \le -\nu;$$

(H5) there are small constants r and R with $0 < r < R$ such that

$$C_2|t|^{\alpha(x)} \le p(x)F(x,t) \le C_3|t|^{p(x)}, \quad \text{for } t \in \mathbb{R} \text{ with } r \le |t| \le R, \text{ a.e. } x \in \Omega, \tag{1.2}$$

where C_2 and C_3 are constants with $0 < C_2 < C_3 < 1$, $\alpha(x) \in C(\bar{\Omega})$ and $1 < \alpha(x) < p(x)$. Moreover, there exists $C_4 > 0$ such that

$$F(x,t) \ge -C_4|t|^{p_+} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{1.3}$$

The assumption (H2) implies that the problem (1.1) is superlinear at infinity. A lot of works concerning the superlinear elliptic boundary value problem have been done by using the usual Ambrosetti–Rabinowitz condition, which is originally due to Ambrosetti and Rabinowitz for the case $p = 2$ in [14], that is,
(AR): there exist $\mu > p_+$ and $M > 0$ such that

$$0 < p_+F(x,t) \le f(x,t)t \quad \text{for all } x \in \Omega \text{ and } |t| \ge M. \tag{1.4}$$

From (1.4) it follows that for some $a, b > 0$

$$F(x,t) \ge a|t|^{\mu} - b \quad \text{for } (x,t) \in \Omega \times \mathbb{R}. \tag{1.5}$$

Obviously, (1.5) implies the much weaker condition (H2).

Let us consider the following function (for simplicity we drop the x-dependence):

$$f(x,t) = |t|^{p_+-2}t\left(p_+\log(1+|t|) + \frac{|t|}{1+|t|}\right), \tag{1.6}$$

then $F(x,t) = |t|^{p_+}\log(1+|t|)$. Then f does not satisfy the (AR) condition for any $\mu > p_+$, but it satisfies our conditions (H2) and (H3). Furthermore, we can show that the function fulfills all hypotheses (H1)–(H5).

In [15], Tan and Fang studied (1.1) under the assumptions (H1), (H2), (H3), (1.3) and
(F): There exists $\delta > 0$ such that $F(x,t) \le 0$ for all $x \in \Omega$ and $|t| \le \delta$.

Using Morse theory, they obtained the existence of one nontrivial weak solution in $W_0^{1,p(x)}(\Omega)$. However, the function $f(x,t)$ in (1.6) does not satisfy the condition (F). Liu and Su [16] used Morse theory and local linking to study the existence of multiple nontrivial solutions for p-Laplacian equations under the following assumption: