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Non-uniform continuity of the flow map for an evolution equation modeling shallow water waves of moderate amplitude

Nilay Duruk Mutlubaş [a], Anna Geyer [b], Bogdan-Vasile Matioc [c,*]

[a] Istanbul Kemerburgaz University, School of Arts and Sciences, Department of Basic Sciences, Mahmutbey Dilmenler Caddesi, No.: 26, 34217 Bağcılar, Istanbul, Turkey

[b] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Facultat de Ciències, 08193 Bellaterra, Barcelona, Spain

[c] Institut für Angewandte Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

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ABSTRACT

We prove that the flow map associated to a model equation for surface waves of moderate amplitude in shallow water is not uniformly continuous in the Sobolev space H^s with $s > 3/2$. The main idea is to consider two suitable sequences of smooth initial data whose difference converges to zero in H^s, but such that neither of them is convergent. Our main theorem shows that the exact solutions corresponding to these sequences of data are uniformly bounded in H^s on a uniform existence interval, but the difference of the two solution sequences is bounded away from zero in H^s at any positive time in this interval. The result is obtained by approximating the solutions corresponding to these initial data by explicit formulae and by estimating the approximation error in suitable Sobolev norms.
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1. Introduction and the main result

We consider a model equation for surface waves of moderate amplitude in shallow water

$$u_t + u_x + 6uu_x - 6u^2u_x + 12u^3u_x + u_{xxx} - u_{xxt} + 14uu_{xxx} + 28u_xu_{xx} = 0, \tag{1}$$

which arises as an approximation of the Euler equations in the context of homogeneous, inviscid gravity water waves. In recent years, several nonlinear models have been proposed in order to understand some important aspects of water waves, like wave breaking or solitary waves. One of the most prominent examples is the Camassa–Holm (CH) equation [1], which is an integrable, infinite-dimensional Hamiltonian system [2–4]. The relevance of the CH equation as a model for the propagation of shallow water waves was discussed by Johnson [5], where it is shown that it describes the horizontal component of the velocity field at a certain depth within the fluid; see also [6]. Building upon the ideas presented in [5], Constantin and Lannes [7] have recently derived the evolution equation (1) as a model for the motion of the free surface of the wave, and they evince that (1) approximates the governing equations to the same order as the CH equation. Besides deriving (1), the authors of [7] also establish the local well-posedness results for the Cauchy problem associated to (1). Relying on a semigroup approach due to Kato [8], Duruk [9] has shown that this feature holds for a larger class of initial data, as well as for solutions which are spatially periodic [10]. The well-posedness in the context of Besov spaces together with the regularity and the persistence properties of strong solutions are studied in [11].

* Corresponding author. Tel.: +49 17680231002.
E-mail addresses: nilay.duruk@kemerburgaz.edu.tr (N. Duruk Mutlubaş), annageyer@mat.uab.cat (A. Geyer), matioc@ifam.uni-hannover.de (B.-V. Matioc).

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Similarly to the CH equation, cf. [12,13], the model equation (1) can also capture the phenomenon of wave breaking: for certain initial data the solution remains bounded, but its slope becomes unbounded in finite time cf. [7,10]. Unlike for the CH equation, which is known to possess global solutions, cf. [12,14], it is not apparent how to control the solutions of (1) globally, due to the fact that this equation involves higher order nonlinearities in u and its derivatives than the CH equation. On the other hand, if one passes to a moving frame, it can be shown that there exist solitary traveling wave solutions decaying at infinity [15]. Their orbital stability has been recently studied in [16] using an approach proposed by Grillakis, Shatah and Strauss [17], which takes advantage of the Hamiltonian structure of (1).

In the present paper, we consider the Cauchy problem associated to (1) in the setting of periodic functions. From the local well-posedness results [9,10], we know that its solutions depend continuously on their corresponding initial data in Sobolev spaces H^s with $s > 3/2$. Our main result states that this dependence is not uniformly continuous. This property was only recently shown to hold true for the CH equation [18,19], and was subsequently confirmed also for the Euler equations [20] and for several related hyperbolic problems such as the $\mu-b$ equation [21], the hyperelastic rod equation [22], for a modified CH system [23], and for the modified CH equation [24]. The main difficulty we encounter compared to all these references is that, as mentioned before, our equation has a higher degree of nonlinearity. Nevertheless, we were able two find two sequences of smooth initial data whose difference converges to zero in H^s, but such that none of them is convergent, with the corresponding solutions of (1) being uniformly bounded on a common (nonempty) interval of existence. Approximating these solutions by explicit formulae, we then successively estimate the error in suitable Sobolev norms and use well-known interpolation properties of the Sobolev spaces and commutator estimates to show that at any time of the common existence interval the difference of the two sequences of exact solutions is bounded from below in the H^s-norm by a positive constant. More precisely, denoting by $u(\cdot; u_0)$, the unique solution of (1) corresponding to the initial data $u_0 \in H^s(\mathbb{S})$ with $s > 3/2$, cf. Theorem 2.1, our main result states:

Theorem 1.1 (*Non-uniform Continuity of the Flow Map*)**.** *For $s > 3/2$, the flow map*

$$u_0 \mapsto u(\cdot; u_0) : H^s(\mathbb{S}) \to C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}))$$

associated to the evolution equation (1) is continuous, but it is not uniformly continuous. More precisely, there exist two sequences of solutions

$$(u_n)_n, (\widetilde{u}_n)_n \subset C([0, T_u], H^s(\mathbb{S})) \cap C^1([0, T_u], H^{s-1}(\mathbb{S})),$$

where $T_u > 0$, and a positive constant $C > 0$ with the following properties:

$$\sup_{n\in\mathbb{N}} \max_{[0,T_u]} \|u_n(t)\|_{H^s} + \|\widetilde{u}_n(t)\|_{H^s} \le C,$$

$$\lim_{n\to\infty} \|u_n(0) - \widetilde{u}_n(0)\|_{H^s} = 0,$$

but

$$\liminf_{n\to\infty} \|u_n(t) - \widetilde{u}_n(t)\|_{H^s} \ge C^{-1}|\sin(t)| \quad \text{for } t \in (0, T_u].$$

The structure of the paper is as follows: In Theorem 2.1 we recall some properties concerning the well-posedness of (1) from [10] and determine a lower bound on the existence time of the solution in H^s in terms of the initial data. Then, we introduce two sequences of approximate solutions $(u^{\omega,n})_n$, $\omega \in \{-1, 1\}$, and compute the approximation error in Lemma 3.1. The corresponding solutions $u_{\omega,n}$ of (1) determined by the initial data $u^{\omega,n}(0)$ are then shown to be uniformly bounded on a common interval of existence, the absolute error $\|u^{\omega,n} - u_{\omega,n}\|$ being computed in different Sobolev norms, cf. Lemmas 4.1–4.3. We end the paper with the proof of the main result.

Notation. Throughout this paper, we shall denote by C positive constants which may depend only upon s. Furthermore, $H^r := H^r(\mathbb{S})$, with $r \in \mathbb{R}$, is the L_2-based Sobolev space on the circle $\mathbb{S} := \mathbb{R}/\mathbb{Z}$. Given $r \in \mathbb{R}$, we let $\Lambda^r := (1 - \partial_x^2)^{r/2}$ denote the Fourier multiplier with symbol $((1 + |k|^2)^{r/2})_{k\in\mathbb{Z}}$. It is well-known that $\Lambda^r : H^q(\mathbb{S}) \to H^{q-r}(\mathbb{S})$ is an isometric isomorphism for all $q, r \in \mathbb{R}$. Furthermore, the Banach space $W^m_\infty := W^m_\infty(\mathbb{S})$, $m \in \mathbb{N}$, consisting of all bounded functions which possess bounded weak derivatives of order less than or equal to n, is endowed with the usual norm.

Some useful estimates. The following commutator estimates play a crucial role in our analysis:

$$\|[\Lambda^r, f]g\|_{L_2} \le C_r \left(\|f_x\|_{L_\infty}\|\Lambda^{r-1}g\|_{L_2} + \|\Lambda^r f\|_{L_2}\|g\|_{L_\infty}\right) \quad \text{for all } r > 3/2, \tag{2}$$

$$\|[\Lambda^\sigma \partial_x, f]g\|_{L_2} \le C\|f\|_{H^s}\|g\|_{H^\sigma} \quad \text{for } s > 3/2 \text{ and } 1 + \sigma \in [0, s]. \tag{3}$$

They hold for all functions $f, g \in C^\infty(\mathbb{S})$ and for the commutator $[S, T] := ST - TS$. The Calderon–Coifman–Meyer estimate (3) follows from Proposition 4.2 in Taylor [25]. The estimate (2) is due to Kato and Ponce [26,27]. Additionally, we shall use the following multiplier inequality

$$\|fg\|_{H^t} \le C\|f\|_{H^t}\|g\|_{H^r} \quad \text{for } t \le r,\ r > 1/2 \tag{4}$$

and $f \in H^t(\mathbb{S})$, $g \in H^r(\mathbb{S})$, cf. e.g. [28].

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