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Pullback attractors for the multi-layer quasi-geostrophic equations of the ocean

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ABSTRACT

This article studies the pullback asymptotic behavior of solutions for a non-autonomous multi-layer quasi-geostrophic model in a two-dimensional domain. We prove the existence of pullback attractors $\mathcal{A}^V$ in V (the velocity has the H^1-regularity) and $\mathcal{A}^H$ in H (the velocity has the L^2-regularity). Then we verify the regularity of the pullback attractors by proving that $\mathcal{A}^V = \mathcal{A}^H$, which implies the pullback asymptotic smoothing effect of the model in the sense that the solutions eventually become more regular than the initial data. The method used in this article is similar to the one used in Zhao and Zhou (2007) in the case of the non-autonomous incompressible non-Newtonian fluid in a two-dimensional domain. Let us mention that the non-homogeneous boundary conditions (and the non-local constraint) present in the multi-layer quasi-geostrophic model makes the estimates more complicated, see Bernier (1994). These difficulties are overcome using the new formulation presented in Tachim Medjo (2009).

1. Introduction

The asymptotic behavior of dynamical systems is a challenging and interesting problem, since it can provide useful information on the future evolution of a system. When the equation is dissipative all solutions converge as $t \to \infty$ to the global attractor. The concept of the attractor has been proved an extremely useful tool for studying the long-term behavior of solutions of a wide variety of dynamical systems [1–13]. The global attractor embodies the large-time dynamics of the equations, corresponding to all sorts of regimes, including the turbulent ones. Although this set may be fairly complicated, in general it has a final dimension [1]. For many dissipative equations including the 2D Navier–Stokes (NS) equations, the existence of a maximal attractor has been proved and their regularity studied (see [1]).

The study of non-autonomous dynamical systems is an important subject and has been paid much attention as evidenced by the references cited in [14–23,11,10]. In [24], the author considers some special classes of non-autonomous dynamical systems and studies the existence and uniqueness of uniform attractors. In [17], the authors present a general approach that is well suited to construct the uniform attractors of some equations arising in mathematical physics, see also [1,25,17]. In this approach, instead of considering a single process associated to the dynamical system, the authors consider a family of processes depending on a parameter (symbol) in some Banach space. The approach preserves the leading concept of invariance, which implies the structure of the uniform attractors. Some problems related to the homogenization and the averaging of uniform global attractors for the NS equations have been analyzed in [26].

With the development of non-autonomous and random dynamical systems, a new type of attractor, called pullback attractor was formulated and investigated in [27,10]. As summarized in [28,29], it consists of a parameterized family of nonempty compact subsets of the state space. Pullback attraction describing this attractor to a component subset for a fixed

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parameter value is achieved by starting progressively earlier in time, that is, at parameter value that is carried forward to the fixed value. Usually, non-autonomous dynamical systems can be formulated in terms of a cocycle mapping for the dynamics in the state space. If the cocycle is continuous with respect to a group σ, which itself is continuous, then the non-autonomous dynamical system can be reduced to a semigroup via the skew-product flow. Results on global attractors for autonomous semi-dynamical systems can thus be adapted to such non-autonomous dynamical systems, [17,30,24]. Recently, using the concept of measure of non-compactness (see [31]), the authors of [32] (see also [22,21,18–20]) derived some necessary and sufficient conditions for the existence of pullback attractors of non-autonomous dynamical systems. The result was later improved in [28], where the authors proposed a sufficient condition for the existence of pullback attractors for norm-to-weak continuous cocycles (that is, cocycles mapping convergent into weakly convergent sequences) in a Banach space.

Geophysical flows such as the ocean and the atmosphere are subjected to various time-dependent or non-autonomous wind forcing. Although steady wind stress has been proved useful in numerical simulations, it is important to study the effect of non-autonomous wind forcing on geophysical flows. In this article, we study the pullback asymptotic behavior of solutions for a non-autonomous multi-layer quasi-geostrophic model in a two-dimensional domain using the result of [29]. We prove the existence of pullback attractors $\mathcal{A}^V$ in V (the velocity has the H^1-regularity) and $\mathcal{A}^H$ in H (the velocity has the L^2-regularity). Then we verify the regularity of the pullback attractors by proving that $\mathcal{A}^V = \mathcal{A}^H$, which implies the pullback asymptotic smoothing effect of the model in the sense that the solutions eventually become more regular than the initial data. Let us recall that the QG model is developed for the simulation of large-scale geophysical currents in the middle latitudes. The usual 3D NS equations are simplified using the hypothesis that the Coriolis force is one of the dominating terms. The simplifications lead to the fact that the internal frictional forces of the fluid are taken proportional to the horizontal Laplacian of the vorticity [3,33,34]. The model assumes that the ocean is divided into N layers. The kth layer with its thickness H_k is coupled with the $(k+1)$th and $(k-1)$th layers through the nonlinear term. Let us mention that the non-homogeneous boundary conditions (and the non-local constraint) present in the multi-layer quasi-geostrophic model makes the estimates more complicated, [3]. These difficulties are overcome using the new formulation presented in [35].

The article is divided as follows. In the next section, we recall from [35] the non-autonomous multi-layer quasi-geostrophic model and its mathematical setting. We also derive some a priori estimates when the external forcing is time dependent. In Section 3, we recall from [29] preliminaries on pullback attractors for cocycle. Then, Section 4 studies the existence of pullback attractors in $\mathcal{A}^V$ using the result of [28]. In Section 5, we prove the existence of pullback attractors $\mathcal{A}^H$ in H when the external force is normal. Then in Section 6, using a method of [29] we verify the regularity of the pullback attractors by proving that $\mathcal{A}^V = \mathcal{A}^H$, which implies the pullback asymptotic smoothing effect of the model in the sense that the solutions eventually become more regular than the initial data.

2. The multi-layer QG equations and their mathematical setting

2.1. Governing equations

We consider the following initial–boundary value problem involving the scalar functions $\tilde{\omega}_k$, $k = 1, 2, \ldots, N$, where N, a positive integer, is the number of layers:

$$\begin{cases} \dfrac{\partial \tilde{\omega}_k}{\partial t} + J(\tilde{\psi}_k, \tilde{\omega}_k + f) + \dfrac{f_0}{H_k}(\tilde{\eta}_k - \tilde{\eta}_{k-1}) - \nu \Delta^2 \tilde{\psi}_k = g_k, \\ \tilde{\eta}_k = \dfrac{f_0}{\tilde{g}_k}\left(\dfrac{\partial}{\partial t}(\tilde{\psi}_{k+1} - \tilde{\psi}_k) + J(\tilde{\psi}_k, \tilde{\psi}_{k+1})\right), \\ \Delta \tilde{\psi}_k = \tilde{\omega}_k. \end{cases} \tag{2.1}$$

The unknown functions are the vorticity $\tilde{\omega}_k$ and the stream function $\tilde{\psi}_k$ in the kth layer. The forcing $g = (g_1, g_2, \ldots, g_N)$ is given. The kth layer of the ocean is characterized by its average height $H_k > 0$ (the actual height is $H_k + \tilde{\eta}_k - \tilde{\eta}_{k-1}$) and its reduced gravity $\tilde{g}_k > 0$. The parameter $f = f_0 + \beta_0 y$ is called the Coriolis parameter, $f_0 > 0$, $\beta_0 > 0$ are physical constants and J is the Jacobian operator defined by

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}. \tag{2.2}$$

The kth layer is coupled with the $(k+1)$th (resp. $(k-1)$th) layer by the term $\tilde{\eta}_k$ (resp. $\tilde{\eta}_{k-1}$) which represents the height of the perturbation from rest of the interface between the kth and the $(k+1)$th (resp. the $(k-1)$th and the kth) layers [36,37]. For simplicity, we introduce the terms $\tilde{\eta}_0$ and $\tilde{\eta}_N$ which are equal to zero. The domain occupied by the fluid is a portion of the cylinder above Ω, where Ω is a bounded open set of $\mathbb{R}^2$ of class $\mathcal{C}^2$ with a boundary $\partial\Omega$. Setting $\tilde{\eta}_0 = H_0 = 0$, the kth layer corresponds to the region $\tilde{\eta}_{k-1} + H_{k-1} < z < \tilde{\eta}_k + H_k$.

The boundary conditions are

$$\begin{cases} \Delta \tilde{\psi}_k = 0 & \text{on } \partial\Omega, \\ \tilde{\psi}_k(t, \cdot) = \tilde{C}_k(t) & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where the $\tilde{C}_k(t) \in \mathbb{R}$ are unknown constants.