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Asymptotic limits and optimal control for the Cahn–Hilliard system with convection and dynamic boundary conditions

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A R T I C L E I N F O

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Dedicated to our friend Prof. Dr. Pierluigi Colli on the occasion of his 60th birthday with best wishes

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A B S T R A C T

In this paper, we study initial–boundary value problems for the Cahn–Hilliard system with convection and nonconvex potential, where dynamic boundary conditions are assumed for both the associated order parameter and the corresponding chemical potential. While recent works addressed the case of viscous Cahn–Hilliard systems, the 'pure' nonviscous case is investigated here. In its first part, the paper deals with the asymptotic behavior of the solutions as time approaches infinity. It is shown that the ω-limit of any trajectory can be characterized in terms of stationary solutions, provided the initial data are sufficiently smooth. The second part of the paper deals with the optimal control of the system by the fluid velocity. Results concerning existence and first-order necessary optimality conditions are proved. Here, we have to restrict ourselves to the case of everywhere defined smooth potentials. In both parts of the paper, we start from corresponding known results for the viscous case, derive sufficiently strong estimates that are uniform with respect to the (positive) viscosity parameter, and then let the viscosity tend to zero to establish the sought results for the nonviscous case.

1. Introduction

In the recent paper [20], the following initial–boundary value problem for the Cahn–Hilliard system with convection was studied,

$$\partial_t \rho + \nabla \rho \cdot u - \Delta \mu = 0 \quad \text{and} \quad \tau_\Omega \partial_t \rho - \Delta \rho + f'(\rho) = \mu \quad \text{in } Q_T := \Omega \times (0, T), \tag{1.1}$$

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where the unknowns ρ and μ represent the order parameter and the chemical potential, respectively, in a phase separation process taking place in an incompressible fluid contained in a container $\Omega \subset \mathbb{R}^3$. In the above equations, τ_Ω is a nonnegative constant, f' is the derivative of a double-well potential f, and u represents the (given) fluid velocity, which is assumed to satisfy $\operatorname{div} u = 0$ in the bulk and $u \cdot \nu = 0$ on the boundary, where ν denotes the outward unit normal to the boundary $\Gamma := \partial\Omega$. Typical and physically significant examples of f are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given, in this order, by

$$f_{reg}(r) := \frac{1}{4}\,(r^2-1)^2\,, \quad r \in \mathbb{R}, \tag{1.2}$$

$$f_{log}(r) := \big((1+r)\ln(1+r) + (1-r)\ln(1-r)\big) - c_1 r^2\,, \quad r \in (-1,1), \tag{1.3}$$

$$f_{2obs}(r) := -c_2 r^2 \quad \text{if } |r| \le 1 \quad \text{and} \quad f_{2obs}(r) := +\infty \quad \text{if } |r| > 1. \tag{1.4}$$

Here, the constants c_i in (1.3) and (1.4) satisfy $c_1 > 1$ and $c_2 > 0$, so that f_{log} and f_{2obs} are nonconvex. In cases like (1.4), one has to split f into a nondifferentiable convex part (the indicator function of $[-1,1]$ in the present example) and a smooth perturbation. Accordingly, one has to replace the derivative of the convex part by the subdifferential and interpret the second identity in (1.1) as a differential inclusion.

As far as the conditions on the boundary Γ are concerned, instead of the classical homogeneous Neumann boundary conditions, the dynamic boundary condition for both μ and ρ were considered, namely,

$$\partial_t\rho_\Gamma + \partial_\nu\mu - \Delta_\Gamma\mu_\Gamma = 0 \quad \text{and} \quad \tau_\Gamma\partial_t\rho_\Gamma + \partial_\nu\rho - \Delta_\Gamma\rho_\Gamma + f'_\Gamma(\rho_\Gamma) = \mu_\Gamma$$
$$\text{on } \Sigma_T := \Gamma \times (0,T), \tag{1.5}$$

where μ_Γ and ρ_Γ are the traces on Σ_T of μ and ρ, respectively; moreover, ∂_ν and Δ_Γ denote the outward normal derivative and the Laplace–Beltrami operator on Γ, τ_Γ is a nonnegative constant, and f'_Γ is the derivative of another potential f_Γ.

The associated total free energy of the phase separation process is the sum of a bulk and a surface contribution and has the form

$$\begin{aligned}
&\mathcal{F}_{\text{tot}}[\mu(t),\mu_\Gamma(t),\rho(t),\rho_\Gamma(t)] \\
&:= \int_\Omega \Big(f(\rho(x,t)) + \frac{1}{2}\,|\nabla\rho(x,t)|^2 - \mu(x,t)\rho(x,t)\Big)\,dx \\
&+ \int_\Gamma \Big(f_\Gamma(\rho_\Gamma(x,t)) + \frac{1}{2}\,|\nabla_\Gamma\rho_\Gamma(x,t)|^2 - \mu_\Gamma(x,t)\rho_\Gamma(x,t)\Big)\,d\Gamma\,,
\end{aligned} \tag{1.6}$$

for $t \in [0,T]$. Moreover, we remark that the Cahn–Hilliard type system (1.5) indicates that on the boundary Γ another phase separation process is occurring that is coupled to the one taking place in the bulk. It is worth noting that the total mass of the order parameter is conserved during the separation process; indeed, integrating the first identity in (1.1) for fixed $t \in (0,T]$ over Ω, using the fact that $\operatorname{div} u = 0$ in Ω and $u \cdot \nu = 0$ on Γ, and invoking the first of the boundary conditions (1.5), we readily find that

$$\partial_t\Big(\int_\Omega \rho(t) + \int_\Gamma \rho_\Gamma(t)\Big) = 0\,. \tag{1.7}$$

The quoted paper [20] was devoted to the study of the initial–boundary value problem obtained by complementing (1.1) and (1.5) with the initial condition $\rho(0) = \rho_0$, where ρ_0 is a given function on Ω. By just assuming that the viscosity coefficients τ_Ω and τ_Γ are nonnegative and that the potentials fulfill suitable assumptions and compatibility conditions, well-posedness and regularity results were established. Moreover, in [12], the study of the longtime behavior was addressed in the viscous case, i.e., if $\tau_\Omega > 0$ and $\tau_\Gamma > 0$. More precisely, the ω-limit (in a suitable topology) of any trajectory (ρ,ρ_Γ) was characterized in terms of