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Riesz potentials and pointwise estimates of solutions to anisotropic porous medium equation☆

Kateryna O. Buryachenko[a,*], Igor I. Skrypnik[b,a]

[a] *Vasyl' Stus Donetsk National University, Mathematical Analysis and Differential Equations, 600-richcha Str., 21, 21021 Vinnytsia, Ukraine*
[b] *Institute of Applied Mathematics and Mechanics, NAS of Ukraine, G. Batiouk Str., 19, 84100 Sloviansk, Ukraine*

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For an anisotropic porous medium equation with L^1 regular lower order terms we prove the local boundedness and continuity of weak solutions in terms of the linear Riesz potential of the right-hand side of equation.

1. Introduction and main results

In this paper we consider nonlinear parabolic equations

$$u_t - \sum_{i=1}^{n} \left(u^{m_i-1}u_{x_i}\right)_{x_i} = f, \text{ in } \Omega_T,\, u \geq 0, \tag{1.1}$$

$$1-\frac{2}{n} < m_1 \leq m_2 \leq \cdots \leq m_{n-1} < m_n < m + \frac{2}{n},\; m = \frac{1}{n}\sum_{i=1}^{n} m_i, \tag{1.2}$$

where $\Omega_T = \Omega \times (0,T)$, Ω is a bounded domain in $\mathbb{R}^n$, $0 < T < \infty$, $n \geq 2$, $f \in L^1_{loc}(\Omega_T)$.

In the isotropic case, i.e. $m_1 = m_2 = \cdots = m_n = m$, this class of equations has numerous applications and has been attracting attention for several decades (see, e.g. the monographs [1,12,37] and references

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* Corresponding author.
E-mail addresses: katarzyna_@ukr.net (K.O. Buryachenko), iskrypnik@iamm.donbass.com (I.I. Skrypnik).

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therein). However, there are few papers on the general case of Eq. (1.1), although it has strong physical background. In fact, it comes from fluid dynamics in anisotropic media. If the conductivities of the media are different in the different directions, the constants m_i in (1.1) must be different from each other (see [2] for the details). Conditions (1.2) are sharp as there are unbounded solutions to (1.1) in Ω_T if the conditions are violated [16,30]. More generally, we deal with the parabolic equation of the type

$$u_t - div\mathbf{A}(x,\, t,\, u,\, \nabla u) = f, \text{ in } \Omega_T, \tag{1.3}$$

where the vector-field $\mathbf{A} = (a_1,\, a_2,\, \dots,\, a_n) : \Omega_T \times \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is Lebesgue measurable with respect to $(x,\, t) \in \Omega_T$ for all $(u,\, \xi) \in \mathbb{R}^1 \times \mathbb{R}^n$, and continuous with respect to $(u,\, \xi)$ for a.a. $(x,\, t) \in \Omega_T$. We also assume that the following structure conditions are satisfied with some positive constants K_1, K_2:

$$\mathbf{A}(x,\, t,\, u,\, \xi)\xi \geq K_1 \sum_{i=1}^{n} u^{m_i-1}|\xi_i|^2,\; u \geq 0,$$
$$|a_i(x,\, t,\, u,\, \xi)| \leq K_2 u^{m_i-1}|\xi|,\; i = 1,\, 2,\, \dots,\, n. \tag{1.4}$$

For the function f we assume that it belongs to $L^1_{loc}(\Omega_T)$. The constants K_1, K_2, $m_1, \dots, m_n$ and n are further referred to as the data. As already mentioned before, throughout the paper we consider the case when some m_i can be less than 1 (so called “singular” case) and the other m_i can be greater than 1 (so called “degenerate” case). The aim of this paper is to establish basic qualitative properties such as local boundedness of weak solutions and their continuity in terms of a linear anisotropic Riesz potential of the right-hand side function f.

Before formulating the main results, let us recall the definition of a weak solution to Eq. (1.3). Set $m^- = \min(m_1,\, 1)$, we say that u is a nonnegative weak solution to Eq. (1.3) if $u \in C^o_{loc}(0,\, T; L^{1+m^-}(\Omega))$ and $u^{\frac{m_i+m^-}{2}-1}u_{x_i} \in L^2(0,\, T;\, L^2(\Omega))$, $i = 1,\, 2,\, \dots, n$ and for every compact $E \subset \Omega$ and for every subinterval $[t_1,\, t_2] \subset (0,\, T]$ the following identity

$$\int_E u\,\varphi\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_E \{-u\,\varphi_t + A(x,\, t,\, u,\, \nabla u)\nabla\varphi\} dx\, dt =$$
$$\int_{t_1}^{t_2}\int_E f\,\varphi\, dx\, dt, \tag{1.5}$$

holds true for any testing function $\varphi \in L^2_{loc}(0,\, T; \overset{o}{W}{}^{1,2}(E))$, φ, $\varphi_t \in L^\infty(E_T)$, $E_T := E \times (0,\, T)$.

The assumption that the testing function φ and its first derivative φ_t must be bounded has to be satisfied in order to guarantee that the terms involving the time derivative and the right-hand side of (1.5) are well defined. All other integrals are finite due to the assumptions on u and φ.

For $v \in L^1(E_T)$ we define the following mollification in time

$$[v]_h(\cdot,\, t) := \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} v(\cdot,\, s) ds, \tag{1.6}$$

for $h \in (0,\, T]$, $t \in (0,\, T]$ and its time reverse analogue

$$[v]_{\bar{h}}(\cdot,\, t) := \frac{1}{h}\int_t^T e^{\frac{t-s}{h}} v(\cdot,\, s) ds,$$

for $h \in (0,\, T]$, $t \in (0,\, T]$. For the properties of this mollification we refer the reader to [7]. If u is a nonnegative weak solution to Eq. (1.3), then its time regularization $[u]_h$ satisfies

$$\iint_{E_T} \left\{\frac{\partial}{\partial t}[u]_h\varphi + [A(x,t,u,\nabla u)]_h\nabla\varphi\right\} dxdt =$$
$$\iint_{E_T} f[\varphi]_{\bar{h}} dxdt,$$

for any $\varphi \in L^2(0,T,W^{1,2}(E)) \bigcap L^\infty(E_T)$ with compact support in E_T [4].