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The Emden–Fowler equation on a spherical cap of $\mathbb{S}^n$

Atsushi Kosaka[a], Yasuhito Miyamoto[b,*]

[a] Bukkyo University, 96, Kitahananobo-cho, Murasakino, Kita-ku, Kyoto 603-8301, Japan
[b] Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

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ABSTRACT

Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, $n \geq 3$, be the unit sphere, and let $S_\Theta \subset \mathbb{S}^n$ be a geodesic ball with geodesic radius $\Theta \in (0, \pi)$. We study the bifurcation diagram $\{(\Theta, \|U\|_\infty)\} \subset \mathbb{R}^2$ of the radial solutions of the Emden–Fowler equation on S_Θ

$$\begin{cases} \Delta_{\mathbb{S}^n} U + U^p = 0 & \text{in } S_\Theta, \\ U = 0 & \text{on } \partial S_\Theta, \\ U > 0 & \text{in } S_\Theta, \end{cases}$$

where $p > 1$. Among other things, we prove the following: For each $p > p_S := (n-2)/(n+2)$, there exists $\underline{\Theta} \in (0, \pi)$ such that the problem has a radial solution for $\Theta \in (\underline{\Theta}, \pi)$ and has no radial solution for $\Theta \in (0, \underline{\Theta})$. Moreover, this solution is unique in the space of radial functions if Θ is close to π. If $p_S < p < p_{JL}$, then there exists $\Theta^* \in (\underline{\Theta}, \pi)$ such that the problem has infinitely many radial solutions for $\Theta = \Theta^*$, where

$$p_{JL} = \begin{cases} 1 + \dfrac{4}{n - 4 - 2\sqrt{n-1}} & \text{if } n \geq 11, \\ \infty & \text{if } 2 \leq n \leq 10. \end{cases}$$

Asymptotic behaviors of the bifurcation diagram as $p \to \infty$ and $p \downarrow 1$ are also studied.

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1. Introduction and main results

Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, $n \geq 3$, be the unit sphere, and let $S_\Theta \subset \mathbb{S}^n$ be the geodesic ball centered at the North Pole with geodesic radius $\Theta \in (0, \pi)$. We call S_Θ the spherical cap. In this paper we are concerned with the solution of the Emden–Fowler equation on S_Θ

$$\begin{cases} \Delta_{\mathbb{S}^n} U + U^p = 0 & \text{in } S_\Theta, \\ U = 0 & \text{on } \partial S_\Theta, \\ U > 0 & \text{in } S_\Theta, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: a-kosaka@bukkyo-u.ac.jp (A. Kosaka), miyamoto@ms.u-tokyo.ac.jp (Y. Miyamoto).

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where $\Delta_{\mathbb{S}^n}$ denotes the Laplace–Beltrami operator on $\mathbb{S}^n$ and $p > 1$. In the Euclidean case it is well known that the qualitative property of the structure of the solutions of the problem

$$\begin{cases} \Delta U + U^p = 0 & \text{in } B_\Lambda, \\ U = 0 & \text{on } \partial B_\Lambda, \\ U > 0 & \text{in } B_\Lambda \end{cases} \tag{1.2}$$

depends on p, and does not depend on Λ. Here, $B_\Lambda \subset \mathbb{R}^n$ denotes the ball centered at the origin O with radius $\Lambda > 0$. By the symmetry result of Gidas, et al. [18], every solution of (1.2) is radially symmetric. The critical Sobolev exponent

$$p_{\mathrm{S}} := \begin{cases} \dfrac{n+2}{n-2}, & \text{if } n \geq 3, \\ \infty, & \text{if } n = 1, 2 \end{cases}$$

plays an important role. It is known that (1.2) has a unique solution if $1 < p < p_{\mathrm{S}}$, and has no solution if $p \geq p_{\mathrm{S}}$ (See Pohoźaev [31]). In the hyperbolic space the moving plane method is applicable and every positive solution of a semilinear elliptic equation with general nonlinearity on a geodesic ball with radius $\Lambda > 0$ is radially symmetric. See [24,32] for this symmetry result. Bonforte, et al. [10] showed, among other things, that in the hyperbolic space the Emden–Fowler equation on the geodesic ball with radius $\Lambda > 0$ has a unique positive solution if $1 < p < p_{\mathrm{S}}$, and has no solution if $p \geq p_{\mathrm{S}}$. Thus, the hyperbolic case is qualitatively the same as the Euclidean case. In the spherical case Padilla [30] and Kumaresan–Prajapat [24] showed that if S_Θ is included in a hemisphere ($0 < \Theta < \frac{\pi}{2}$), then every positive solution of a semilinear elliptic equation with general nonlinearity is radially symmetric. On the other hand, if S_Θ includes a hemisphere ($\frac{\pi}{2} < \Theta < \pi$), then Bandle–Wei [5] constructed nonradial positive solutions of the problem

$$\begin{cases} \Delta_{\mathbb{S}^n} U + \ell U + U^p = 0 & \text{in } S_\Theta, \\ U = 0 & \text{on } \partial S_\Theta \end{cases} \tag{1.3}$$

when ℓ is negatively large. See [1,3,5,6,8,9,11,16,26] for other solutions of (1.3). As far as (1.1) is concerned, if $0 < \Theta < \pi$ and $1 < p \leq p_{\mathrm{S}}$, then one can easily show that the solution is radial, changing variables and applying the symmetry result of [18] to the equation. When $\Theta = \frac{\pi}{2}$ and $p > 1$, the radial symmetry of a solution of (1.1) is guaranteed by [32, Theorem 1]. The question whether a solution of (1.1) is radial in the case where $\frac{\pi}{2} < \Theta < \pi$ and $p > p_{\mathrm{S}}$ seems to remain open. In this paper we restrict ourselves to radially symmetric solutions.

This study is motivated by the result of Bandle–Peletier [4]. In the case where $n = 3$ and $p = p_{\mathrm{S}} (= 5)$ they showed that (1.1) has no solution if S_Θ is included in a hemisphere, and has a radial solution if S_Θ includes a hemisphere. This indicates that the solution structure depends not only on p but also on the radius Θ. Actually, we will see in Corollary B that (1.1) has a solution even in the supercritical case $p > p_{\mathrm{S}}$ if Θ is close to π. Hence, the solution structure in the spherical case is different from the solution structures in both the Euclidean and hyperbolic cases. The problem (1.1) can be considered as a special case of (1.3). The problem (1.3) also appears in the Dirichlet problem of the Emden–Fowler equation $\Delta u + u^p = 0$ in a cone of $\mathbb{R}^{n+1}$ of base S_Θ. Let $u(x) := |x|^{-2/(p-1)} U(x/|x|)$. Then U satisfies (1.3) with $\ell = -\frac{2}{p-1}\left(n - 1 - \frac{2}{p-1}\right)$. In particular, Bidaut-Véron, et al. [8] obtained existence, non-existence and uniqueness results for (1.3) and Corollary 2.1 and Remark 2.1 of [8] cover the case where $\ell = 0$ and p is in a part of the supercritical range $p > p_{\mathrm{S}}$. The supercritical Emden–Fowler equation on other manifolds was studied in Berchio, et al. [7].

Let us explain the problem in detail. Let θ be the geodesic distance from the North Pole of $\mathbb{S}^n$. Let $p > 1$ be fixed. Then the solution U of (1.1) depends only on θ. The problem (1.1) can be reduced to the ODE

$$\begin{cases} U'' + (n-1)\dfrac{\cos\theta}{\sin\theta} U' + U^p = 0, & 0 < \theta < \Theta, \\ U(\Theta) = 0, & \\ U > 0, & 0 \leq \theta < \Theta. \end{cases} \tag{1.4}$$

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