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Principal eigenvalue and maximum principle for cooperative periodic–parabolic systems

I. Antón[a], J. López-Gómez[b,*]

[a] Departamento de Análisis Matemático y Matemática Aplicada, Universidad Complutense de Madrid, Madrid 28040, Spain
[b] Instituto de Matemática Interdisciplinar (IMI), Departamento de Análisis Matemático y Matemática Aplicada, Universidad Complutense de Madrid, Madrid 28040, Spain

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ABSTRACT

This paper classifies the set of supersolutions of a general class of periodic–parabolic problems in the presence of a positive supersolution. From this result we characterize the positivity of the underlying resolvent operator through the positivity of the associated principal eigenvalue and the existence of a positive strict supersolution. Lastly, this (scalar) characterization is used to characterize the strong maximum principle for a class of periodic–parabolic systems of cooperative type under arbitrary boundary conditions of mixed type.

1. Introduction

This paper gives a periodic–parabolic counterpart of the second classification theorem of J. López-Gómez [18] and infers from it a periodic–parabolic counterpart of [17, Th. 2.5] and Theorem 2.4 of H. Amann and J. López-Gómez [6]. Then, based on that result, the main theorem of [7], which was originally stated for cooperative systems subject to Dirichlet boundary conditions, is substantially sharpened up to cover the case of general boundary operators of mixed type. The elliptic counterparts of these results have shown to be a milestone for the generation of new results in spatially heterogeneous nonlinear elliptic equations and cooperative systems (see, e.g., P. Álvarez-Caudevilla and J. López-Gómez [1,2], M. Molina-Meyer [24–26], H. Amann [5], J. López-Gómez and L. Maire [21] and the recent monograph [20]). Thus, the findings of

* Corresponding author.
E-mail addresses: iantonlo@ucm.es (I. Antón), Lopez_Gomez@mat.ucm.es (J. López-Gómez).

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this paper seem imperative for analyzing a wide variety of problems in the presence of spatio temporal heterogeneities.

In this paper we are working under the following general assumptions:

(A1) Ω is a bounded subdomain (open and connected set) of $\mathbb{R}^N$, $N \geq 1$, of class $\mathcal{C}^{2+\theta}$ for some $0 < \theta \leq 1$, whose boundary, $\partial\Omega$, consists of two disjoint open and closed subsets, Γ_0 and Γ_1, respectively, such that $\partial\Omega := \Gamma_0 \cup \Gamma_1$ (as they are disjoint, Γ_0 and Γ_1 must be of class $\mathcal{C}^{2+\theta}$). It must be stressed that $\partial\Omega$ cannot be connected if both $\Gamma_i \neq \emptyset$, $i = 0,1$.

(A2) For a given $T > 0$, we consider the non-autonomous differential operator

$$\mathfrak{L} := \mathfrak{L}(x,t) := -\sum_{i,j=1}^{N} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{N} b_j(x,t)\frac{\partial}{\partial x_j} + c(x,t) \tag{1.1}$$

with $a_{ij} = a_{ji}, b_j, c \in F$ for all $1 \leq i,j \leq N$, where

$$F := \left\{ u \in \mathcal{C}^{\theta,\frac{\theta}{2}}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}) \ : \ u(\cdot, T+t) = u(\cdot, t) \ \text{ for all } t \in \mathbb{R} \right\}. \tag{1.2}$$

Moreover, we assume that $\mathfrak{L}$ is uniformly elliptic in $\bar{Q}_T$, where Q_T stands for the parabolic cylinder

$$Q_T := \Omega \times (0,T),$$

i.e., there exists $\mu > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geq \mu\,|\xi|^2 \quad \text{for all } \ (x,t,\xi) \in \bar{Q}_T \times \mathbb{R}^N,$$

where $|\cdot|$ stands for the Euclidean norm of $\mathbb{R}^N$.

(A3) $\mathfrak{B} : \mathcal{C}(\Gamma_0) \oplus \mathcal{C}^1(\Omega \cup \Gamma_1) \to C(\partial\Omega)$ stands for the boundary operator

$$\mathfrak{B}\xi := \begin{cases} \xi & \text{on} \quad \Gamma_0 \\ \dfrac{\partial \xi}{\partial \nu} + \beta(x)\xi & \text{on} \quad \Gamma_1 \end{cases} \tag{1.3}$$

for each $\xi \in \mathcal{C}(\Gamma_0) \oplus \mathcal{C}^1(\Omega \cup \Gamma_1)$, where $\beta \in \mathcal{C}^{1+\theta}(\Gamma_1)$ and

$$\nu = (\nu_1, \ldots, \nu_N) \in \mathcal{C}^{1+\theta}(\partial\Omega; \mathbb{R}^N)$$

is an outward pointing nowhere tangent vector field.

Thus, rather crucially, in this paper the function β can change sign, in strong contrast with the classical setting dealt with by P. Hess [13] and, more recently, by R. Peng and X. Q. Zhao [28], where it was imposed the strongest condition $\beta \geq 0$. In our general setting, $\mathfrak{B}$ is the *Dirichlet boundary operator* on Γ_0, and the *Neumann*, or a *first order regular oblique derivative boundary operator*, on Γ_1, and either Γ_0, or Γ_1, can be empty. As in this paper β can change of sign, our results can be applied straight away to deal with cooperative periodic–parabolic systems under general *nonlinear mixed boundary conditions*.

Besides the space F introduced in (1.2), this paper also considers the Banach spaces of Hölder continuous T-periodic functions

$$E := \left\{ u \in \mathcal{C}^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}) \ : \quad u(\cdot, T+t) = u(\cdot, t) \ \text{ for all } \ t \in \mathbb{R} \right\}$$

and the periodic–parabolic operator

$$\mathcal{P} := \partial_t + \mathfrak{L}(x,t). \tag{1.4}$$