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Multiplicity results for variable-order fractional Laplacian equations with variable growth

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ABSTRACT

In this paper, we study the multiplicity of solutions for an elliptic type problem driven by the variable-order fractional Laplace operator involving variable exponents. More precisely, we consider

$$\begin{cases} (-\Delta)^{s(\cdot)}u + \lambda V(x)u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $N \geq 1$, $s(\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ is a continuous function, Ω is a bounded domain in $\mathbb{R}^N$ with $N > 2s(x,y)$ for all $(x,y) \in \Omega \times \Omega$, $(-\Delta)^{s(\cdot)}$ is the variable-order fractional Laplace operator, $\lambda > 0$ is a parameter, $V : \Omega \to [0,\infty)$ is a continuous function, $\alpha, \beta > 0$ are two parameters and $p, q \in C(\Omega)$. Under some suitable assumptions, we show that the above problem admits at least two distinct solutions by applying the mountain pass theorem and Ekeland's variational principle. Then we prove that these two solutions converge to two solutions of a limit problem as $\lambda \to \infty$. Moreover, we obtain the existence of infinitely many solutions for the limit problem.

1. Introduction and main results

In this paper we study an elliptic equation involving variable exponent driven by the fractional Laplace operator with variable order derivative. More precisely, we consider

$$\begin{cases} (-\Delta)^{s(\cdot)}u + \lambda V(x)u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $s(\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ is a continuous function, $N > 2s(x,y)$ for all $(x,y) \in \Omega \times \Omega$, $(-\Delta)^{s(\cdot)}$ is the variable-order fractional Laplace operator, $\lambda > 0$ is a parameter, $V : \Omega \to [0,\infty)$ is a continuous function

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and $\alpha, \beta > 0$ are two parameters and $p, q \in C(\Omega)$. The variable-order fractional Laplace operator $(-\Delta)^{s(\cdot)}$ is defined as follows: for each $x \in \mathbb{R}^N$,

$$(-\Delta)^{s(\cdot)}\varphi(x) = 2P.V. \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s(x,y)}} dy,$$

along any $\varphi \in C_0^\infty(\Omega)$, where $P.V.$ denotes the Cauchy principle value. Since $s(\cdot)$ is a function, operator $(-\Delta)^{s(\cdot)}$ is called variable order fractional Laplace operator. Especially, when $s(\cdot) \equiv constant$, $(-\Delta)^{s(\cdot)}$ reduce to the usual fractional Laplace operator.

The fractional variable order derivatives suggested by Lorenzo and Hartley in [23] appeared in nonlinear diffusion processes. In particular, some diffusion processes reacting to temperature changes may be better described by using variable order derivatives in a nonlocal integro-differential operator, see for example [24] for more details. Leopold in [21] considered multifractional pseudodifferential models in the representation of heterogeneous local behaviors and studied the solutions to such models in fractional Besov spaces of variable order in $\mathbb{R}^N$. Kikuchi and Negoro in [18] obtained the conditions that general pseudodifferential operators on fractional Sobolev spaces of variable order in $\mathbb{R}^N$ form a Feller semigroup which has a transition density. Especially, Ruiz-Medina et al. in [31] studied some Gaussian processes defined by elliptic pseudodifferential equations, in which the covariance function of these random processes defines the inner product of a fractional Sobolev space of variable order.

Especially, when $s(\cdot) \equiv s(constant)$, then the nonlocal integro-differential operator $(-\Delta)^{s(\cdot)}$ reduces to the usual fractional Laplacian, see [9,12,22] and the references therein for more details with respect to the fractional Laplacian. In recent years, a great attention has been focused on the study of nonlocal fractional operators and related fractional differential equations. This type of operators arises in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcome of stochastic stabilization of Lévy processes, see [3,8,20] and the references therein. The literature on fractional Laplace operators and their applications is very interesting and quite large, we refer the interested readers to [4,15,16,27–30,33–38] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the readers to [12].

If $s(\cdot) \equiv 1$, then problem (1.1) becomes the well-known second order elliptic equation with variable growth conditions

$$\begin{cases} -\Delta u + \lambda V(x)u = \alpha |u|^{p(x)-2}u + \beta |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.2}$$

Such kinds of model can be used to describe various phenomena. In [10], Chen et al. proposed a framework for image restoration based on a variable exponent Laplacian. Another main application which involved variable exponent Laplace operators was related to the modeling of electrorheological fluids, see [32]. For more results on variable exponent problems, for instance, see [1,13,25,26].

It is worth mentioning that there are some papers concerning related equations involving the fractional $p(x)$-Laplace operator. In fact, results for the fractional Sobolev spaces with variable exponent and fractional $p(x)$-Laplace equations are few, for example, we refer to [5,6,17]. Here we point out that fractional $p(x)$- Laplace operator involves the constant case of $s(\cdot)$ on the available literature.

Inspired by the above works, we assume that $s : \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ and $V : \Omega \to [0,\infty)$ are two continuous functions satisfying

- (S_1) $0 < s^- := \min_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y) \leq s^+ := \max_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y) < 1$.
- (S_2) $s(\cdot)$ is symmetric, that is, $s(x,y) = s(y,x)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.
- (V_1) $J = \text{int}(V^{-1}(0)) \subset \Omega$ is a nonempty bounded domain and $\widetilde{J} = V^{-1}(0)$;
- (V_2) there exists a nonempty open domain $\Omega_0 \subset J$ such that $V(x) \equiv 0$ for all $x \in \overline{\Omega_0}$.