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Eigenvalue estimates for a class of elliptic differential operators in divergence form

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We compute estimates for eigenvalues of a class of linear second-order elliptic differential operators in divergence form (with Dirichlet boundary condition) on a bounded domain in a complete Riemannian manifold. Our estimates are based upon the Weyl's asymptotic formula. As an application, we find a lower bound for the mean of the first k eigenvalues of the drifting Laplacian. In particular, we have extended for this operator a partial solution given by Cheng and Yang for the generalized conjecture of Pólya. We also derive a second-Yang type inequality due to Chen and Cheng, and other two inequalities which generalize results by Cheng and Yang obtained for a domain in the unit sphere and for a domain in the projective space.

1. Introduction

In this paper $(M, \langle , \rangle)$ is a complete Riemannian manifold and the domain $\Omega \subset M$ is connected bounded with smooth boundary $\partial\Omega$ in M. Let T be a symmetric positive definite $(1,1)$-tensor on M and $\eta \in \mathcal{C}^2(M)$. We are interested in studying the eigenvalue problem with Dirichlet boundary condition, namely:

$$\begin{cases} -\mathscr{L}u & = & \lambda u & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega, \end{cases}$$

where

$$\mathscr{L}u = \operatorname{div}(T(\nabla u)) - \langle \nabla\eta, T(\nabla u)\rangle. \tag{1.1}$$

In this more general setting, we apply known techniques to prove our first result.

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Theorem 1. *Let Ω be a domain in an n-dimensional complete Riemannian manifold M isometrically immersed in $\mathbb{R}^m$, λ_i be the ith eigenvalue of $\mathscr{L}$ and u_i its corresponding normalized real-valued eigenfunction. Then*

$$\operatorname{tr}(T)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\Big((m-n)^2A_0^2T_*^2+(T_0+T_*\eta_0)^2 + 4(T_0+T_*\eta_0)\|T(\nabla u_i)\|_{L^2(\Omega,\mathrm{dm})}+4\lambda_i\Big),$$

where $A_0=\max\{\sup_{\bar\Omega}|A_{e_k}|,\ k=n+1,\dots,m\}$, A_{e_k} is the Weingarten operator of the immersion with respect to e_k, $\eta_0=\sup_{\bar\Omega}|\nabla\eta|$, $T_*=\sup_{\bar\Omega}|T|$ and $T_0=\sup_{\bar\Omega}|\operatorname{tr}(\nabla T)|$.

The drifting Laplacian case $L=\Delta-\langle\nabla\eta,\nabla\cdot\rangle$ is recovered when T is the identity operator in $\mathfrak{X}(M)$. In Section 4 we work specifically in this case. The main results for L are described in the following two theorems.

Theorem 2. *Let Ω be a domain in an n-dimensional complete Riemannian manifold M isometrically immersed in $\mathbb{R}^m$ with mean curvature H, and λ_i be the ith eigenvalue of the drifting Laplacian. Then*

$$\frac{1}{k}\sum_{i=1}^{k}\upsilon_i \ge \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \mathrm{vol}\,\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \text{for} \quad k=1,2,\dots$$

where $\upsilon_i:=\lambda_i+\frac{n^2H_0^2+\eta_0^2+2\bar\eta_0}{4}$, $\eta_0=\sup_{\bar\Omega}|\nabla\eta|$, $\bar\eta_0=\sup_{\bar\Omega}|L\eta|$ and $H_0=\sup_{\bar\Omega}\|H\|$.

Theorem 2 is an extension for L of Theorem 1.1 in Cheng and Yang [8]. In particular, we have extended for the drifting Laplacian a partial solution given by them to the generalized conjecture of Pólya (see Conjecture 1).

Theorem 3. *Under the assumptions in Theorem 2, we have*

$$\upsilon_{k+1} \le \frac{1}{k}\Big(1+\frac{4}{n}\Big)\sum_{i=1}^{k}\upsilon_i, \tag{1.2}$$

$$\upsilon_{k+1} \le \Big(1+\frac{2}{n}\Big)\frac{1}{k}\sum_{i=1}^{k}\upsilon_i + \Big[\Big(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\upsilon_i\Big)^2 - \Big(1+\frac{4}{n}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\upsilon_j-\frac{1}{k}\sum_{i=1}^{k}\upsilon_i\Big)^2\Big]^{\frac{1}{2}}, \tag{1.3}$$

$$\upsilon_{k+1}-\upsilon_k \le 2\Big[\Big(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\upsilon_i\Big)^2 - \Big(1+\frac{4}{n}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\upsilon_j-\frac{1}{k}\sum_{i=1}^{k}\upsilon_i\Big)^2\Big]^{\frac{1}{2}}. \tag{1.4}$$

Estimate (1.2) is a second-Yang type inequality due to Chen and Cheng [4, Inequality (1.5)], whereas the other two estimates generalize results by Cheng and Yang [5,6] obtained for a domain in the unit sphere $\mathbb{S}^n(1)$ and for a domain in the projective space $\mathbb{CP}^n(4)$. More precisely, inequalities (1.3) and (1.4) generalize Theorem 1 and Corollary 1 in [5], respectively, as well as Theorem 1 and Corollary 1 in [6].

2. Motivating the definition of the operator $\mathscr{L}$

In this section, we establish the necessary tools to work with the operator defined in (1.1) which enable us to obtain more general results. We believe this operator would be also useful in obtaining rigidity results or characterizing some known geometric objects.