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Elliptic gradient estimates for the doubly nonlinear diffusion equation

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ABSTRACT

In this paper we establish elliptic type gradient estimates for positive solutions to the doubly nonlinear diffusion equation on compact or complete noncompact Riemannian manifolds. As special cases, we can derive elliptic type gradient estimates for positive solutions to the p-Laplace parabolic equations and the fast diffusion equations. Harnack inequalities derived from these elliptic type gradient estimates can be used to compare solutions at the same time.

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1. Introduction

Let M be an n-dimensional Riemannian manifold with metric g. In this paper we study the following doubly nonlinear diffusion equation

$$\partial_t u = \triangle_p(u^\gamma) = \mathrm{div}(|\nabla(u^\gamma)|^{p-2}\nabla(u^\gamma)), \tag{1.1}$$

where we assume that $\gamma > 0$, $p > 1$ and $\triangle_p$ is the p-Laplace operator. The doubly nonlinear diffusion equation appears in non-Newtonian fluids, glaciology and turbulent flows in porous media, etc. [3,14,20], which becomes the heat equation, the porous medium equation, the fast diffusion equation, or the p-Laplace heat equation if we choose $p = 2$ and $\gamma = 1$, $p = 2$ and $\gamma > 1$, $p = 2$ and $\gamma < 1$, or $\gamma = 1$, respectively.

Li-Yau [8] established the classical parabolic type gradient estimate for positive solutions to the linear heat equation, and then deduced the Harnack inequalities and the estimates for the heat kernel. Li–Yau's gradient estimate had been generalized to some other settings, for example, on smooth metric measure spaces

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via the Bakry–Émery curvature [9]; on complete Riemannian manifolds with the metric evolving along the Ricci flow [2], on connected locally finite graphs under suitable curvature-dimension conditions [1,17], etc. Parabolic type gradient estimate and the entropy formula for the p-Laplace heat equation were studied in [7].

Gradient estimates for the porous medium equations and fast diffusion equations were studied in [10]. Wang-Chen [20] proved a sharp global parabolic type gradient estimate for positive solutions to the doubly nonlinear diffusion equation on complete Riemannian manifolds with the nonnegative Ricci curvature, as an application, they could obtain a Harnack inequality. Chen–Xiong [3] obtained Li–Xu type, Hamilton type and Davies type gradient estimates for positive solutions to the doubly nonlinear diffusion equation on compact Riemannian manifolds with the Ricci curvature bounded from below by a negative constant, as an application, they also derived corresponding Harnack inequalities. Gradient estimates for the porous medium equation on complete Riemannian manifolds with the metric evolving along the Ricci flow were established in [12,18].

Harnack inequalities deduced from the parabolic type gradient estimate can only be used to compare solutions at different times. In order to overcome this blemish, one can consider the elliptic type gradient estimate. Harnack inequalities deduced from the elliptic type gradient estimate can be used to compare solutions at the same time. Hamilton [5] firstly got the elliptic type gradient estimate on a compact manifold. Souplet–Zhang [13] proved the elliptic type gradient estimate for the linear heat equation on a complete noncompact Riemannian manifold. Elliptic type gradient estimates for positive solutions to the p-Laplace parabolic equation

$$|u|^{p-2}\partial_t u = \triangle_p u \tag{1.2}$$

on complete Riemannian manifolds with metrics fixed or evolving along the Ricci flow were established in [15] or [16] respectively. Elliptic type gradient estimates of the porous medium equations and fast diffusion equations were established in [6,21,22].

In this paper, we will establish elliptic type gradient estimates for positive solutions to (1.1). As stated in Section 2, (1.1) is equivalent to (2.2) or (2.3), where v is defined in (2.1). Since the right hand side of (2.2) is nonlinear, we introduce an operator $\mathcal{L}$, which is linear and positive definite for $p > 1$ at any points where $\nabla v \neq 0$ and $bv > 0$. Based on the Bochner formula we establish an evolving equation of

$$(\mathcal{L} - \partial_t)(\frac{|\nabla v|^p}{(1-v)^p}).$$

By using the maximum principle we establish an elliptic type gradient estimate for positive solutions to (1.1) on compact manifolds. When studying on complete noncompact Riemannian manifolds we need to use the cut-off function and can only consider the case that $1 < p \leq 2$, now the computation involves the full Hessian of the distance function, as a result, a lower bound on the sectional curvature has to be assumed. As byproducts, we can derive Harnack inequalities, which can be used to compare solutions at the same time.

This paper is organized as follows. In Section 2 we give some basic calculation for positive solutions to the doubly nonlinear diffusion equation (1.1). In Section 3 we establish an elliptic type gradient estimate on compact manifolds with the Ricci curvature bounded from below, we also derive the corresponding Harnack inequality which can be used to compare solutions at the same time. A similar result on complete noncompact Riemannian manifolds is proved in Section 4, now we should assume that $1 < p \leq 2$ and the sectional curvature is bounded from below. In Section 5 we study several special cases of the doubly nonlinear diffusion equation. In Section 6 we pose two questions.

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