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Slope estimate and boundary differentiability for inhomogeneous infinity Laplace equation on convex domains

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ABSTRACT

We study the boundary differentiability for inhomogeneous infinity Laplace equations on convex domains Ω with the inhomogeneous term $f \in C(\Omega) \cap L^{\infty}(\Omega)$ and differentiable boundary data g. At a flat point (the boundary point where the blow up of the domain is the half-space), u is differentiable due to a previous result of the second author in Hong (2014). At a corner point (the boundary point where the blow up of the domain is not the half-space), we establish a slope estimate for u, and provide a necessary and sufficient condition for the boundary differentiability of u in this paper.

1. Introduction

The infinity Laplace equation

$$\triangle_{\infty} u(x) := \sum_{1 \leq i,j \leq n} u_{x_i} u_{x_j} u_{x_i x_j} = 0$$

was introduced by G. Aronsson in the 1960s [1] as the Euler–Lagrange equation of the sup-norm variational problem of $|\nabla u|$ or the equivalent absolutely minimizing Lipschitz extension (AMLe) problem. It is a highly degenerate elliptic partial differential equation. In 1993, R. Jensen [13] proved that the Dirichlet problem:

$$\triangle_{\infty} u = 0 \text{ in } \Omega, \ \ u = g \text{ on } \partial\Omega$$

has a unique viscosity solution for any bounded domain $\Omega \subset \mathbb{R}^n$ and $g \in C(\partial\Omega)$. Such a solution u is called an infinity harmonic function. He also proved that a function $u \in C(\Omega)$ is an AMLe if and only if u is a

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viscosity solution to $\triangle_\infty u = 0$. In 2001, Crandall–Evans–Gariepy [3] provided that $u \in C(\Omega)$ is an infinity harmonic function if and only if u enjoys the comparison with cones property.

The regularity problems are always the core issues of the elliptic *pde*s. Evans and Smart [7] proved the interior differentiability of u. This is the best result that has been achieved so far. For dimension 2, Savin [16] and Evans–Savin [6] proved the C^1 and $C^{1,\alpha}$ regularity earlier.

The boundary regularity of u was initially studied by Wang–Yu [17]. They proved the boundary differentiability of u in general dimensions with the assumption that $\partial\Omega$ and g are C^1. They also proved the C^1 boundary regularity of u for dimension 2 if $\partial\Omega$ and g are C^2. The second author of this paper improved Wang–Yu's first result to that if both $\partial\Omega$ and g are differentiable at a boundary point $x_0 \in \partial\Omega$ then u is differentiable at x_0 [8]. In another paper [10], Hong constructed a two dimensional counterexample to show that $|Du|$ may not be continuous along the boundary if $\partial\Omega$ is merely assumed to be C^1.

In [12], Hong–Liu investigated the boundary regularity of u on convex domains. The boundary points of a convex domain are divided into flat points (where the blow up of the domain is the half-space) and corner points (where the blow up of the domain is not the half-space). The boundary of a convex domain is differentiable at a flat point, so u is differentiable at a flat point if g is differentiable at this point by the former result in [8]. The interesting case is the behavior of u at a corner point. Assume that g is differentiable at a corner point x_0. In general, the solution u need not be differentiable at x_0 (see Example 1 in [12]). They proved that the slope function (defined later) $S(x_0) \le |Dg(x_0)|$ and provided a sufficient and necessary condition for the establishment of differentiability of u. Our work in this article is to prove that these two conclusions are also true for the solutions of the inhomogeneous infinity Laplace equation with continuous and bounded right hand side.

Let $\Omega \subset R^n$ be a bounded and connected open set. The Dirichlet problem for inhomogeneous infinity Laplace equation

$$\begin{cases} \triangle_\infty u &= f, \quad in\ \Omega \\ u &= g, \quad on\ \partial\Omega \end{cases} \tag{1}$$

was introduced by Lu–Wang [15]. They achieved the existence and uniqueness of the viscosity solution of (1) under the conditions that $f \in C(\Omega)$ with $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$ and $g \in C(\partial\Omega)$. They also gave the comparison principle and the comparison with standard functions property. These properties were applied to prove the stability of the inhomogeneous infinity Laplace equation with nonvanishing f.

E. Lindgren [14] proved that at an interior point the blow ups of u are linear if $f \in C(\Omega)$ and u is differentiable if $f \in C^1(\Omega)$. If $f \equiv -1$, $g \equiv 0$, Ω is convex and satisfies an interior sphere condition, Crasta–Fragalà [4] achieved the solution u is power-concave and of class C^1, and proved the expected optimal regularity of u is $C^{1,1/3}$. For Serrin-type overdetermined boundary value problems in [5], C^1 regularity continues to hold without the interior sphere condition. In the normalized case with the operator $\triangle_\infty^N$[5], they also showed that stadium-like domains are precisely the unique convex sets in R^n where the solution to a Dirichlet problem is of class $C^{1,1}(\Omega)$.

Now we assume that $u \in C(\Omega)$ is the viscosity solution of (1). Define the slope functions as in [17], for $x \in \bar{\Omega}$,

$$S_r^+(x) = \sup_{y \in \partial(B_r^n(x)\cap\Omega)\setminus\{x\}} \frac{u(y) - u(x)}{|y - x|},$$

$$S_r^-(x) = \sup_{y \in \partial(B_r^n(x)\cap\Omega)\setminus\{x\}} \frac{u(x) - u(y)}{|y - x|},$$

and $S_r(x) := \max\{S_r^+(x), S_r^-(x)\}$, with

$$B_r^n(x) := \{y \in R^n : |y - x| < r\}$$

for $x \in R^n$, $B_r^n := B_r^n(0)$.