Nonlinear Analysis 176 (2018) 56–83

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Existence of parabolic minimizers on metric measure spaces

Michael Collins*, Andreas Herán

Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstrasse 11, 91058 Erlangen, Germany

A R T I C L E I N F O

Article history:
Received 16 January 2018
Accepted 8 June 2018
Communicated by S. Carl

MSC:
49J27
49J40
30L99
35A15

Keywords:
Existence
Parabolic minimizers
Metric measure spaces

A B S T R A C T

The objects of our studies are vector valued parabolic minimizers u associated to a convex Carathéodory integrand f obeying a p-growth assumption from below and a certain monotonicity condition in the gradient variable. Here, the functions being considered are defined on a metric measure space $(\mathcal{X}, d, \mu)$. For such parabolic minimizers that coincide with a time-independent Cauchy–Dirichlet datum u_0 on the parabolic boundary of a space–time-cylinder $\Omega \times (0,T)$ with an open subset $\Omega \subset \mathcal{X}$ and $T > 0$, we prove existence in the parabolic Newtonian space $L^p(0,T;\mathcal{N}^{1,p}(\Omega;\mathbb{R}^N))$. In this paper we generalize results from Bögelein et al. (2014,2015) to the metric setting and argue completely on a variational level.

1. Introduction

We are concerned with the existence of parabolic minimizers on metric measure spaces. More precisely, we consider minimizers of integral functionals that are related to vector-valued functions $u : \Omega \times (0,T) \to \mathbb{R}^N$, $N \geq 1$, which satisfy the inequality

$$\iint u \cdot \partial_t \varphi \,\mathrm{d}\mu\,\mathrm{d}t + \iint f(x,u,g_u)\,\mathrm{d}\mu\,\mathrm{d}t \leq \iint f(x,u+\varphi,g_{u+\varphi})\,\mathrm{d}\mu\,\mathrm{d}t \tag{1.1}$$

for a Carathéodory-integrand $f : \Omega \times \mathbb{R}^N \times [0,\infty] \to \mathbb{R}$. Here, "$\cdot$" denotes the standard scalar-product on $\mathbb{R}^N$ and $\Omega \subset \mathcal{X}$ is a bounded domain, where $(\mathcal{X}, d, \mu)$ is a metric measure space with a metric d and measure μ.

In the setting of a metric measure space, the classical calculus known from the Euclidean space $\mathbb{R}^n$ is no longer available and instead of the gradient Du we have to introduce the notion of upper gradients. The "minimal upper gradient" will be denoted by g_u.

* Corresponding author.
E-mail addresses: collins@math.fau.de (M. Collins), heran@math.fau.de (A. Herán).

https://doi.org/10.1016/j.na.2018.06.005

This paper deals with parabolic minimizers on parabolic cylinders $\Omega_T := \Omega \times (0,T)$ with $\Omega \subset \mathcal{X}$ bounded and open and $T > 0$. $\mathcal{X}$ denotes a metric measure space that fulfills a doubling property with respect to the metric d and the measure μ and supports a suitable Poincaré inequality. We refer to Chapter 2 for exact definitions and the setting of the relevant spaces. In this paper, we are going to generalize results which have recently been proven in [6] and [7], respectively, to this framework.

Since the beginning of the 21st century, doubling metric measure spaces have been studied quite extensively, see for example [9,14,22,23,25,27,37,49,50] and especially [2] for an overview and further references. The idea of considering variational problems on metric measure spaces is based on independent proofs of Grigor'yan [20] and Saloff-Coste [47] of the fact that on Riemannian manifolds the doubling property and Poincaré inequality are equivalent to a certain Harnack-type inequality for solutions of the heat equation. But instead of Riemannian manifolds, we are interested in more general spaces.

Existence for parabolic problems on metric measure spaces has to our knowledge not been dealt with in the past. In the elliptic case, the Dirichlet problem for p-harmonic functions (i.e. $f(x,u,z) = z^p$) has been considered by Björn & Björn in [2, Chapters 8 & 10] and [3], the latter a joint work with Shanmugalingam. By the results of Cheeger in [9], where it is shown that his definition of Sobolev spaces on metric spaces leads to a reflexive space via the existence of a differential as a measurable section of a finite dimensional cotangent bundle, one can see that direct methods in the calculus of variations can be applied to prove the existence for the p-Dirichlet-problem, see also [50]. The investigation of parabolic problems on metric measure spaces started not long ago with the work of Kinnunen, Marola, Miranda and Paronetto, see [33], concerning regularity problems. Since then, the most contributions in this field of research have been made to stability theory (see e.g. [16,17,40]) and regularity problems (see also [15,21,43,45,44]). When concerning regularity problems, one tries to establish e.g. higher integrability or Hölder continuity of a solution that is assumed to be an element of the parabolic Newtonian space $L^p(0,T;\mathcal{N}^{1,p}(\Omega))$. By that space, we denote functions $u : (0,T) \to \mathcal{N}^{1,p}(\Omega)$, such that the mapping $t \mapsto \|u(t)\|^p_{\mathcal{N}^{1,p}(\Omega)}$ is integrable over the interval $(0,T)$.

Our aim is to show existence for a parabolic minimizer of a functional

$$\mathcal{F}(w,\Omega_T) := \iint_{\Omega_T} f(\cdot, w, g_w)\, \mathrm{d}\mu\, \mathrm{d}t$$

in such a parabolic Newtonian space, which will be done via the concept of global variational solutions. The integrand is required to be convex in the second and third variable and to fulfill a coercivity assumption of the form

$$f(x,u,z) \geq \nu z^p - h(x)(1+|u|)$$

for all (x,u,z) in $\Omega \times \mathbb{R}^N \times [0,\infty]$ with a constant $\nu > 0$, $1 < p < \infty$ and $0 \leq h \in L^{p'}(\Omega;\mathbb{R}^N)$, where $p' := \frac{p}{p-1}$ denotes the Hölder conjugate of p. Also, we are going to assume f to be increasing in the z-variable. Again, we refer the reader to Chapter 2 for the exact definitions and in particular to Theorem 2.5 and Corollary 2.6, respectively, to find the exact statements.

In the paper at hand, we are going to put the focus on how to overcome certain difficulties given by the setting of metric measure spaces. The main difficulty that leads to such obstacles is given by the fact that the upper gradient g_u unlike the classical gradient Du does not behave linearly but only in a sub-linear way, such as the modulus $|Du|$ does. In fact, for $\mathcal{X} = \mathbb{R}^n$ there holds $g_u = |Du|$. Therefore, if we are going to generalize results that have been found for integrands $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ in the Euclidean case, we have to think about what happens if the integrand only depends on the modulus of the third variable, i.e. $f(x,u,z) = g(x,u,|z|)$. For the sake of simplicity, we are going to consider the model case $f(z) = g(|z|)$ with a convex function g now. Such integrands have been looked at before in the Euclidean case, see i.e. [10–12,18,41,42]. Additional assumptions have been made there, so are the authors in [18] for