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On a salt fingers model

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ABSTRACT

We consider the model introduced in Paparella and von Hardenberg (2014), that consists in the homogeneous boundary value problem for a system of nonlinear degenerate parabolic equations. We prove the existence of global weak solutions and discuss their stability and asymptotic properties.

1. Introduction

Fingering convection is a mixing phenomenon that occurs when two scalar fields with different diffusivities affect the density of a fluid in a competing way. An instability occurs when the most-diffusing one is stratified so as to make the fluid statically stable, while the stratification of the least-diffusing one counteracts the effect of the first, reducing the magnitude that the vertical density gradient would have in its absence. The typical example of this set-up occurs naturally in the subtropical oceans, where warm, salty water overlies colder and fresher water. There, a water mass displaced downward loses by diffusion its heat to the surrounding cooler fluid, without losing much of its less-diffusive salinity. This results in the water mass becoming heavier than the surrounding fluid, and thus sinking to much greater depths than the original small perturbation

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would have brought it. Conversely, a water mass displaced upward acquires heat without acquiring much salinity, and thus gains the buoyancy required for rising further. This sort of instability may occur in a large variety of systems, such as magma chambers, stellar interiors, industrial crucibles, reacting flows. In all cases it is characterized by a well-defined horizontal spatial scale, which, in the ocean, corresponds to just a few centimeters. For a review of the phenomenon, including the linear analysis of the instability sketched above in words, the interested reader may refer to [18, chap. 8] and to [19].

Fingering convection is thus a form of convective flow produced by small, buoyancy-carrying structures (the so-called salt fingers) in a fluid where the overall density decreases upward (rather than increases, as in the single-scalar Rayleigh–Bénard convection) [21]. Laboratory experiments and, recently, numerical simulations, have revealed the existence of a further instability that occurs when the convection is sufficiently vigorous (see e.g. [15] and references therein). Close to marginality and far from the boundaries, the motion of the salt fingers tends to slowly bring to a constant value the vertical gradient of the horizontally averaged scalar fields, gradually removing small vertical inhomogeneities that may be present in the initial conditions. However, when the convection is vigorous, solutions starting from constant initial gradients develop spontaneously a staircase-looking profile, that alternates regions of steep vertical gradients to nearly homogeneous regions.

An attempt to understand the basic physical processes leading to the formation of these staircase-like structures lead to the heuristic mathematical model presented in [16]. The model consists of the following initial boundary value problem with Neumann homogeneous boundary conditions for two degenerate nonlocal parabolic equations:

$$\begin{cases} \partial_t u = \partial_x(\Phi(u)\partial_x u) + \alpha - \Phi(u)\partial_x v - \beta u\sqrt{K[\partial_x v]}, & t > 0, 0 < x < 1, \\ \partial_t v = \partial_x(\Phi(u)\partial_x v), & t > 0, 0 < x < 1, \\ \partial_x u(t,0) = \partial_x u(t,1) = 0, & t > 0, \\ \partial_x v(t,0) = \partial_x v(t,1) = 0, & t > 0, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), & 0 < x < 1. \end{cases} \tag{1.1}$$

The unknowns

$$u = u(t,x), \qquad v = v(t,x),$$

represent, respectively, the *horizontally averaged kinetic energy per unit mass* and the *horizontally averaged buoyancy*. The spatial variable x represents the vertical extent of the fluid, and is oriented in the upward direction (namely, the local gravity acceleration vector points in the direction of decreasing values of x).

The modeling strategy follows the approach used in [1] to describe a similar instability occurring in mechanically forced, stratified turbulent flows. In particular, small-scale motions are not explicitly described by the model. Turbulent overturnings that tend to produce down-gradient mixing are subsumed under an effective diffusion coefficient Φ, whose magnitude is a function of the local kinetic energy density of the flow. The effect of salt fingers, with extreme simplification, is reduced to the constant buoyancy flux α. Following Prandtl's mixing length theory, the diffusion coefficient is expressed as the product of a characteristic velocity times a characteristic length. It thus assumes the form:

$$\Phi \in C^\infty(\mathbb{R}_+), \quad \text{such that} \quad \Phi(u) = \begin{cases} \sqrt{u} f(u), & u \geq 2, \\ \sqrt{u + \gamma} f(u), & u \leq 1, \end{cases} \tag{1.2}$$

where $\mathbb{R}_+ = [0, \infty)$. The precise prescriptions on the mixing length f embody much of the physical insight that goes into the model. The results of this paper are valid for any choice of the mixing length having the