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The fractional Schrödinger equation with Hardy-type potentials and sign-changing nonlinearities

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ABSTRACT

We look for solutions to a fractional Schrödinger equation of the following form

$$(-\Delta)^{\alpha/2}u + \left(V(x) - \frac{\mu}{|x|^\alpha}\right) u = f(x,u) - K(x)|u|^{q-2}uon\mathbb{R}^N \setminus \{0\},$$

where V is bounded and close-to-periodic potential and $-\frac{\mu}{|x|^\alpha}$ is a Hardy-type potential. We assume that V is positive and f has the subcritical growth but not higher than $|u|^{q-2}u$. If μ is positive and small enough we find a ground state solution, i.e. a critical point of the energy being minimizer on the Nehari manifold. If μ is negative we show that there is no ground state solutions. We are also interested in an asymptotic behaviour of solutions as $\mu \to 0^+$ and$K \to 0$.
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1. Introduction

We consider the following nonlinear, fractional Schrödinger equation with external, Hardy-type potential

$$(-\Delta)^{\alpha/2}u + \left(V(x) - \frac{\mu}{|x|^\alpha}\right) u = f(x,u) - K(x)|u|^{q-2}u \text{ on } \mathbb{R}^N \setminus \{0\} \tag{1.1}$$

where $\alpha \in (0,2)$, $\mu \in \mathbb{R}$ and $N > \alpha$, with $u \in H^{\alpha/2}(\mathbb{R}^N)$. The fractional Schrödinger equation arises in many models from mathematical physics, e.g. nonlinear optics, quantum mechanics, nuclear physics (see e.g. [16,30,33,39,40,46,52,56,58,59] and references therein). We focus on the external potential of the form $V(x) - \frac{\mu}{|x|^\alpha}$, where $V \in L^\infty(\mathbb{R}^N)$ is close-to-periodic potential and $-\frac{\mu}{|x|^\alpha}$ is Hardy-type potential. Note that

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the Hardy-type potential does not belong to the Kato's class, hence it is not a lower order perturbation of the operator $-\Delta + V(x)$ (see [45]).

The fractional Laplacian can be defined via Fourier multiplier $|\xi|^\alpha$, i.e. the operator $(-\Delta)^{\alpha/2}$, for a function $\psi : \mathbb{R}^N \to \mathbb{R}$, is given by the Fourier transform by the formula

$$\mathcal{F}\left((-\Delta)^{\alpha/2}\psi\right)(\xi) := |\xi|^\alpha \hat{\psi}(\xi),$$

where

$$\mathcal{F}\psi(\xi) := \hat{\psi}(\xi) := \int_{\mathbb{R}^N} e^{-i\xi\cdot x}\psi(x)\, dx$$

denotes the usual Fourier transform. When $\psi : \mathbb{R}^N \to \mathbb{R}$ is rapidly decaying smooth function, it can be defined by the principal value of the singular integral

$$(-\Delta)^{\alpha/2}\psi(x) = c_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{\psi(x) - \psi(y)}{|x-y|^{N+\alpha}}\, dy, \tag{1.2}$$

where

$$c_{N,\alpha} := \frac{2^\alpha \Gamma\left(\frac{N+\alpha}{2}\right)}{2\pi^{N/2}|\Gamma(-\alpha/2)|} > 0.$$

Here, Γ denotes the Gamma function, i.e. a function defined for complex numbers z with $\mathrm{Re}(z) > 0$ by the formula

$$\Gamma(z) := \int_0^\infty x^{z-1}e^{-x}\, dx$$

and extended to a meromorphic function on the set $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Both definitions of the fractional Laplacian are equivalent, i.e. on $L^2(\mathbb{R}^N)$ they give operators with common domain and they coincide on this domain (see [35]). It is known that $(-\Delta)^{\alpha/2}$ reduces to $-\Delta$ as $\alpha \to 2^-$ (see [17]). In this paper we identify $(-\Delta)^{\alpha/2}$ with the classical Laplace operator $-\Delta$ for $\alpha = 2$. In what follows we will use the following characterization of the fractional Sobolev space, for $0 < \alpha < 2$:

$$H^{\alpha/2}(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) \ : \ \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\, dx\, dy + \int_{\mathbb{R}^N} |u(x)|^2\, dx < \infty \right\}$$

with the associated scalar product:

$$H^{\alpha/2}(\mathbb{R}^N) \times H^{\alpha/2}(\mathbb{R}^N) \ni (u,v) \mapsto \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+\alpha}}\, dx\, dy + \int_{\mathbb{R}^N} u(x)v(x)\, dx \in \mathbb{R}.$$

See e.g. [10,17] for more background about the fractional Laplace operator and fractional Sobolev spaces.

Eq. (1.1) describes the behaviour of the so-called standing wave solutions $\Phi(x,t) = u(x)e^{-i\omega t}$ of the following time-dependent fractional Schrödinger equation

$$i\frac{\partial \Phi}{\partial t} = (-\Delta)^{\alpha/2}\Phi + \left(V(x) - \frac{\mu}{|x|^\alpha} + \omega\right)\Phi - g(x, |\Phi|).$$

Such an equation was introduced by Laskin by expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths (see [36,37]). The time-dependent equation is also intensively studied (see e.g. [28,38]).

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