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Boundary behavior of k-convex solutions for singular k-Hessian equations

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A B S T R A C T

We discuss the existence and boundary behavior of k-convex solution to the singular k-Hessian problem

$$\begin{cases} S_k(D^2u(x)) = b(x)f(-u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $S_k(D^2u)(k \in \{1, 2, \ldots, n\})$ is the k-Hessian operator, $\Omega \subset R^n(n \geq 2)$ is a smooth bounded strictly convex domain. Here the weight function $b(x)$ is not necessarily bounded on $\partial\Omega$. Another interest is that $f(u) \to \infty$ as $u \to 0$. Our approach mainly relies on Karamata's regular variation theory and the construction of suitable sub- and super-solutions.

1. Introduction

The k-Hessian problem arises in geometric problems, fluid mechanics and other applied subjects. For example, under the case $k = n$, the k-Hessian problem can describe Weingarten curvature, or reflector shape design (see [36]). In recent years, increasing attention has been paid to the study of the k-Hessian problems by different methods (see [3,11,13,14,23,31,33,38,40–42]).

Moreover, for $k \geq 2$ we know that the k-Hessian operator is a fully nonlinear partial differential operator, and we notice that some fully nonlinear elliptic operators have attracted the attention of Harvey and Lawson ([19–22]), Caffarelli, Li and Nirenberg ([4,5]), Amendola, Galise and Vitolo [1], Galise and Vitolo [12], Capuzzo-Dolcetta, Leoni and Vitolo [7], Vitolo [39], Lazer and McKenna [27], Zhang [47], Tso [37], and Zhang and Du [44]. However, in literature there aren't articles on boundary behavior of solution to the singular k-Hessian equation. More precisely, the study of the weight function $b(x)$ is not necessarily bounded on $\partial\Omega$ and $f(u) \to \infty$ as $u \to 0$ are still open for the k-Hessian problem.

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At the same time, we notice that the boundary asymptotic behavior of solutions of singular elliptic problems has attracted the attention of Crandall, Rabinowitz and Tartar [10], Ghergu and Rădulescu [16], Lazer and McKenna [26], Zhang [46], and Zhang and Li [48]. Especially, let us review several excellent results related to our problem of k-Hessian equations. In [29], Loewner and Nirenberg considered the existence of solution for the Monge–Ampère problem

$$\begin{cases} \det D^2u = u^{-(n+2)} \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $n = 2$. In [8], Cheng and Yau studied problem (1.1) in a more general case $n \geq 2$ and obtained the existence results of problem (1.1).

In [46], Lazer and McKenna presented a unique result for the Monge–Ampère problem

$$\begin{cases} \det D^2u = b(x)u^{-\gamma} \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

where $\gamma > 1$ and $b \in C^\infty(\bar{\Omega})$ is positive. Applying regularity theory and sub–supersolution method, they got a unique solution u satisfying $u \in C^2(\Omega) \cap C(\bar{\Omega})$, and they proved that there exist two negative constants c_1 and c_2, such that u satisfies

$$c_1 d(x)^\beta \leq u(x) \leq c_2 d(x)^\beta \ \text{ in } \Omega,$$

where $\beta = \frac{n+1}{n+\gamma}$ and $d(x) = dist(x, \partial\Omega)$.

Recently, Mohammed [30] established the existence and the global estimates of solutions of the Monge–Ampère problem:

$$\begin{cases} \det D^2u = b(x)f(-u(x)) \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset R^n (n \geq 2)$, $f \in C^\infty(0, \infty)$ is positive and decreasing, and $b \in C^\infty(\Omega)$ is positive in Ω.

Very recently, Li and Ma [28] studied the existence and the boundary asymptotic behavior of solutions of problem (1.3) by using regularity theory and sub–supersolution method. An overview of the asymptotic behavior of solutions of elliptic problems can be found in Ghergu and Radulescu [15].

For all we know, in literature there aren't articles on the existence and boundary behavior of solutions to k-Hessian equations, especially for the singular k-Hessian problems. More precisely, the study of $S_k(D^2u)(k \in \{1, 2, \ldots, n\})$, the weight function $b(x)$ is not necessarily bounded on $\partial\Omega$ and $f(u) \to \infty$ as $u \to 0$ are still open for the following problem:

$$\begin{cases} S_k(D^2u(x)) = b(x)f(-u(x)), \ x \in \Omega, \\ u(x) = 0, \ x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset R^n (n \geq 2)$ is a smooth bounded strictly convex domain, f is singular at zero, and $S_k(D^2u)(k \in \{1, 2, \ldots, n\})$ is the k-Hessian operator. It denotes the kth elementary symmetric function of the eigenvalues of D^2u, the Hessian of u, i.e.

$$S_k(D^2u) = S_k(\lambda_1, \lambda_2 \ldots, \lambda_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k},$$

where $\lambda_1, \lambda_2 \ldots, \lambda_n$ are the eigenvalues of D^2u (see [6,32]).

On the other hand, it is commonly known that $\{S_k : k \in \{1, 2, \ldots, n\}\}$ contains many well known operators, such as the Laplace operator ($k = 1$), and the Monge–Ampère operator ($k = n$).