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On the Cauchy problem for a nonlinear variational wave equation with degenerate initial data

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ABSTRACT

This paper is focused on a one-dimensional nonlinear variational wave equation which is the Euler–Lagrange equations of a variational principle arising in the theory of nematic liquid crystals and a few other physical contexts. We establish the local existence and uniqueness of classical solutions to its Cauchy problem with initial data given on the parabolic degenerating line.

1. Introduction

The variational principle whose action is a quadratic function of the derivatives of the field with coefficients depending on the field and the independent variables reads that [2,16,17]

$$\delta \int A^{ij}_{\mu\nu}(\mathbf{x},u)\frac{\partial u^\mu}{\partial x_i}\frac{\partial u^\nu}{\partial x_j}\mathrm{d}\mathbf{x} = 0, \tag{1.1}$$

where the summation convention is employed. Here, $\mathbf{x} \in \mathbb{R}^{d+1}$ are the space–time independent variables and $\mathbf{u} : \mathbb{R}^{d+1} \to \mathbb{R}^n$ are the dependent variables, the coefficients $A^{ij}_{\mu\nu} : \mathbb{R}^{d+1} \times \mathbb{R}^n \to \mathbb{R}$ are smooth functions.

A particular motivation for studying the variational principle (1.1) comes from the theory of nematic liquid crystals. In nematic liquid crystals, the average orientation of molecules can be described by the so-called director field $\mathbf{n}(\mathbf{x},t) \in \mathbb{S}^2$ at a spatial location $\mathbf{x}$ and time t. The potential energy density for the

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director field $\mathbf{n}$ is given by the well-known Frank–Oseen energy density

$$W(\mathbf{n},\nabla\mathbf{n})=\frac{1}{2}k_1(\nabla\cdot\mathbf{n})^2+\frac{1}{2}k_2(\mathbf{n}\cdot\nabla\times\mathbf{n})^2+\frac{1}{2}k_3|\mathbf{n}\times(\nabla\times\mathbf{n})|^2,$$

Here k_1,k_2 and k_3 are the splay, twist and bend elastic constants of the material, respectively. In the regime in which inertia effects dominate viscosity, Saxton [21] modeled the propagation of the orientation waves in the director field $\mathbf{n}$ by the least action principle

$$\delta\int\left(\frac{1}{2}\partial_t\mathbf{n}\cdot\partial_t\mathbf{n}-W(\mathbf{n},\nabla\mathbf{n})\right)\mathrm{d}\mathbf{x}\mathrm{d}t=0,\qquad \mathbf{n}\cdot\mathbf{n}=1, \tag{1.2}$$

which is a special case of variational principle (1.1). For the planar deformations of $\mathbf{n}$ that depend only on a single space variable x with $\mathbf{n}=(\cos u(t,x),\sin u(t,x),0)$, the Euler–Lagrange equation of (1.2) is

$$u_{tt}-c(u)[c(u)u_x]_x=0\quad\text{with}\;\; c^2(u)=k_1\sin^2u+k_3\cos^2u. \tag{1.3}$$

See [16,21,31] for the details on the derivation and physical background of the above equation. Generally, the elastic constants k_1 and k_3 are positive and then the wave speed $c(\cdot)$ is a strictly positive function. However, in some cases, see e.g. [1,9,20], the elastic constant k_1 or k_3 may be negative which implies that the wave speed $c(\cdot)$ can be zero. For another application, if $c(u)=u$, then (1.3) reduces to the second sound equation in one space dimension

$$u_{tt}-u(uu_x)_x=0, \tag{1.4}$$

see Kato and Sugiyama [18] for details.

Under the assumption that the wave speed $c(\cdot)$ is a positive function, the nonlinear variational wave equation (1.3) has been widely explored. Even for smooth initial data, Glassey, Hunter and Zheng [10] have shown that the solution can lose regularity in finite time due to the nonlinear nature. There are many authors considering the global existence of two natural distinct classes of weak solutions (dissipative and conservative) to its initial data problem. In a series of papers [23–26], Zhang and Zheng have studied carefully the global existence of dissipative solutions for (1.3) and its asymptotic models by using the Young measure theory. Bressan and Huang [7] provided a different approach to construct a global dissipative solution. By introducing the method of energy-dependent coordinates, Bressan and Zheng [8] established the global existence of conservative solutions to its Cauchy problem for initial data of finite energy. Holden and Raynaud [11] obtained the existing result by using slightly different language from that in [8]. Recently, the uniqueness of conservative solutions has been proved by Bressan, Chen and Zhang [6]. Moreover, Bressan and Chen [5] addressed the stability of the conservative solutions by constructing a Finsler metric which renders the flow uniformly Lipschitz continuous. See the survey paper by Bressan [4] for the details. In [13], Hu investigated a more general variational wave equation than (1.3) arising from the variational principle (1.1). We also refer the reader to Refs. [3,12,27,28] for the discussion of the global existence of conservative solutions to the one-dimensional nonlinear variational wave systems.

Until recently, studies on the degenerate hyperbolic problems for the nonlinear variational wave equations are still very limited. In a recent paper [15], we investigated the variational wave equation

$$u_{tt}-[c^2(u,x)u_x]_x=-c(u,x)c_u(u,x)u_x^2, \tag{1.5}$$

which arises from the variational principle (1.1) and includes (1.3) as special example, and established the global existence of smooth solutions to a degenerate initial–boundary value problem under relaxed conditions on the initial–boundary data. In addition, the uniform regularity of solution up to the parabolic degenerate boundary and the regularity of the degenerate curve were discussed. In the present paper, we study the local