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A pocket guide to nonlinear differential equations in Musielak–Orlicz spaces

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ABSTRACT

The Musielak–Orlicz setting unifies variable exponent, Orlicz, weighted Sobolev, and double-phase spaces. They inherit technical difficulties resulting from general growth and inhomogeneity.

In this survey we present an overview of developments of the theory of existence of PDEs in the setting including reflexive and non-reflexive cases, as well as isotropic and anisotropic ones. Particular attention is paid to problems with data below natural duality in absence of Lavrentiev's phenomenon.

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1. Introduction

Vast literature describes various aspects of PDEs with the leading part of the operator having a power-type growth with the preeminent example of the p-Laplacian. There is a wide range of directions in which the polynomial growth case has been developed including variable exponent, convex, weighted, and double-phase approaches. Musielak–Orlicz spaces cover all of the mentioned cases. They are equipped with the norm defined by means of

$$\varrho_M(\nabla u) = \int_\Omega M(x, \nabla u)\, dx,$$

called **modular** and $M : \Omega \times \mathbb{R}^N \to [0, \infty)$ is then called a modular function. We call this function non-homogeneous due to its x-dependence. We assume M to be convex with respect to the gradient variable. Note that M can be an anisotropic function, which means that it depends on whole gradient ∇u, not only

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its length $|\nabla u|$ and the behaviour of M may vary in different directions. The typical anisotropic example is $M(x, \nabla u) = \sum_{i=1}^{N} |u_{x_i}|^{p_i(x)}$, but the anisotropy does not have to be expressed by separation of roles of coordinates.

Let us point out a few basic references. We refer to pioneering works by Orlicz [169], where variable exponent spaces are introduced, and by Zygmund [205] with elements of Orlicz spaces. Variable exponent spaces are carefully treated in the monographs by Cruz-Uribe and Fiorenza [67] and Diening, Harjulehto, Hästö, and Ružička [74]. The foundations of Orlicz spaces are described by Krasnosel'skii and Rutickii [131] and Rao and Ren in [176]. The most exhaustive study on weighted Sobolev setting is presented by Turesson in [191].

Passing to general Musielak–Orlicz spaces — the first systematic approach to non-homogeneous setting with general growth was provided by Nakano [166], then Skaff [179,180] and Hudzik [123,124], but the key role in the functional analysis of modular spaces is played by the comprehensive book by Musielak [164]. None of these sources is focused on the theory of PDEs.

We start with a brief presentation of spaces included in the framework of the Musielak–Orlicz setting in connection to PDEs highlighting only samples of vastness of results therein. Our aim is to present difficulties that each of examples carry rather than to provide a comprehensive overview of results in each setting. Moreover, to keep our guide pocket-sized we restrict ourselves to the theory of existence of solutions to problems in classical euclidean spaces slipping over regularity issues comprehensively described in [158] with further developments in variable exponent spaces mentioned in [119]. Particular attention is paid here to the issue of existence to problems with data below natural duality and relate them either to growth conditions, or to the absence of Lavrentiev's phenomenon when asymptotic behaviour of a modular function is sufficiently balanced.

2. Overview of spaces

This section is devoted to concise summary of features and difficulties of spaces included in the framework of the Musielak–Orlicz setting.

2.1. Sobolev spaces: classical, weighted, and anisotropic

The natural setting to study solutions to elliptic and parabolic partial differential equations involving Laplace or p-Laplace operator

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big)$$

is classical Sobolev spaces, when the modular function has a form $M(x, \nabla u) = |\nabla u|^p$. Let us restrict ourselves to the classical references [137,138]. However, if one wants to study more degenerate partial differential problems involving various types of singularities in the coefficients, e.g. the weighted ω-p-Laplacian

$$\Delta_p^{\omega} u = \operatorname{div}\big(\omega(x) |\nabla u|^{p-2} \nabla u\big),$$

the relevant setting is weighted Sobolev spaces, see [91,122], where the modular function is $M(x, \nabla u) = \omega(x) |\nabla u|^p$.

We refer to [132], where Kufner and Opic introduce the assumption sufficient for the weighted space to be continuously embedded in $L^1_{loc}(\Omega)$, and consequently for any function from the weighted space to have well-defined distributional derivatives. This condition is called B_p-condition, and yields that the weight ω is a positive a.e. Borel measurable function such that $\omega' = \omega^{-1/(p-1)}(x) \in L^1_{loc}(\Omega)$. This condition is weaker than A_p-condition, cf. [161]. Turesson's book [191] consists of a comprehensive study on the case

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