Nonlinear Analysis 175 (2018) 28–55

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Weakly coupled systems of semilinear effectively damped waves with time-dependent coefficient, different power nonlinearities and different regularity of the data

Abdelhamid Mohammed Djaouti, Michael Reissig*

Faculty for Mathematics and Computer Science, Technical University Bergakademie, Prüferstr. 9, 09596 Freiberg, Germany

ARTICLE INFO

Article history:
Received 13 November 2017
Accepted 10 May 2018
Communicated by S. Carl

MSC:
35L52
35L71

Keywords:
Weakly coupled hyperbolic systems
Damped wave equations
Cauchy problem
Global existence
L^2-decay
Effective dissipation
Small data solutions

ABSTRACT

We study the global existence of small data solutions to the Cauchy problem for semilinear damped wave equations with an effective dissipation term, where the data are supposed to belong to different classes of regularity. We apply these results to the Cauchy problem for weakly coupled systems of semilinear effectively damped waves with respect to the defined classes of regularity for different power nonlinearities.

1. Introduction

Many papers are concerned with qualitative properties of solutions to the Cauchy problem for the classical damped wave equation without any source or sink. The model we have in mind is

$$u_{tt} - \Delta u + u_t = 0, \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x).$$

Among all properties decay estimates for solutions and their partial derivatives are of special interest. Such estimates were given in [11]. Having such estimates in hand one can study the following Cauchy problem for semilinear classical damped wave equations with power nonlinearity

$$u_{tt} - \Delta u + u_t = |u|^p, \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x),$$

* Corresponding author.
E-mail addresses: djaouti_abdelhamid@yahoo.fr (A. Mohammed Djaouti), reissig@math.tu-freiberg.de (M. Reissig).

https://doi.org/10.1016/j.na.2018.05.006
0362-546X/

where $p > 1$. In [12] the authors proved for given compactly supported initial data $(u_0, u_1) \in H^1 \times L^2$ and for $p \leq p_{GN}(n) = \frac{n}{n-2}$ if $n \geq 3$ the local (in time) existence of energy solutions $u \in \mathcal{C}([0,T), H^1) \cap \mathcal{C}^1([0,T), L^2)$. In the same paper the global (in time) existence was proved for small data by using the technique of potential well and modified potential well. The authors proposed the critical exponent $p_{crit} = p_{crit}(n) = 1 + \frac{4}{n}$ which means that we have global (in time) existence of small data weak solutions for some admissible $p > p_{crit}$, while local (in time) existence for $p > 1$ and large data. In the paper [19] assuming compactly supported data $(u_0, u_1) \in H^1 \times L^2$ to be sufficiently small, the authors proved a global existence result for $p > p_{Fuj}(n) = 1 + \frac{2}{n}$ and $p \leq p_{GN}(n)$ if $n \geq 3$.

Later some papers are devoted to weakly coupled systems of semilinear classical damped wave equations with power nonlinearities

$$\begin{aligned} &u_{tt} - \Delta u + u_t = |v|^p, \;\; v_{tt} - \Delta v + v_t = |u|^q, \\ &u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x). \end{aligned} \tag{1}$$

In [18] the authors have shown the existence and nonexistence of energy solutions to (1), provided that the space dimension $n = 1, 3$. The interplay between the power nonlinearities is presented by the following condition:

$$\alpha := \frac{\max\{p; q\} + 1}{pq - 1} < \frac{n}{2}. \tag{2}$$

If this condition is satisfied, then to given small data there exists a global (in time) energy solution which satisfies some decay estimates. On the contrary, if this condition is not satisfied, then every energy solution to given initial data having positive average value does not exist globally. In [13] these existence results were generalized to the cases $n = 1, 2, 3$. Improved time decay estimates were given in the case $n = 3$, still under the condition (2). Recently, in [14] the authors determined the critical exponent for any space dimensions. Here "critical" means that if $\alpha < \frac{n}{2}$, then the local (in time) energy solution can be extended globally for suitably chosen small data and if $\alpha \geq \frac{n}{2}$, then suitable local (in time) energy solutions blow up in finite time. The proof of the global (in time) existence of solutions for supercritical nonlinearities is based on a weighted energy method of solutions for supercritical nonlinearities provided that (2) still holds.

Let us now include a time-dependent coefficient $b = b(t)$ in the dissipation term. Then we are interested in the Cauchy problem

$$u_{tt} - \Delta u + b(t) u_t = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \;\; (t,x) \in [0,\infty) \times \mathbb{R}^n. \tag{3}$$

In [20] and [21] a classification of dissipation terms $b(t)u_t$ in scattering to free waves producing, non-effective dissipation terms, effective dissipation terms and overdamping producing are proposed. In the following we only consider effective dissipation terms. Here $b(t)u_t$ is called effective in the model (3) if $b = b(t)$ satisfies the following properties:

- b is a positive and monotonic function with $tb(t) \to \infty$ as $t \to \infty$,
- $((1+t)^2 b(t))^{-1} \in L^1(0,\infty)$,
- $b \in \mathcal{C}^3[0,\infty)$ and $|b^{(k)}(t)| \lesssim \frac{b(t)}{(1+t)^k}$ for $k = 1, 2, 3$,
- $\frac{1}{b} \notin L^1(0,\infty)$ and there exists a constant $a \in [0,1)$ such that $tb'(t) \leq ab(t)$.

Typical examples are

$$b(t) = \frac{\mu}{(1+t)^r}, \;\; b(t) = \frac{\mu}{(1+t)^r} (\log(e+t))^\gamma, \;\; b(t) = \frac{\mu}{(1+t)^r (\log(e+t))^\gamma}$$

for some $\mu > 0$, $\gamma > 0$ and $r \in (-1,1)$. In order to treat semilinear Cauchy problems of the form

$$u_{tt} - \Delta u + b(t) u_t = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x) \tag{4}$$