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Periodic patterns for a model involving short-range and long-range interactions

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A B S T R A C T

We consider a physical model where the total energy is governed by surface tension and repulsive screened Coulomb potential on the 3-dimensional space. We obtain different periodic equilibrium patterns i.e. stationary sets for this energy, under some volume constraints. The patterns bifurcate smoothly from straight lamellae, lattices of round solid cylinders and lattices of round balls.

1. Introduction and main results

In this paper, we are interested with a class of periodic stationary sets in $\mathbb{R}^3$ for the following energy functional

$$\mathcal{P}_\gamma(\Omega) := |\partial\Omega| + \gamma \int_\Omega \int_\Omega G_\kappa(|x-y|)dxdy, \tag{1.1}$$

where $\kappa, \gamma > 0$ and $G_\kappa(r) = \frac{1}{r}e^{-\kappa r}$ is the repulsive Yukawa potential (or the *screened* repulsive Coulomb potential). The Yukawa potential $\overline{G}_\kappa(x) = G_\kappa(|x|)$ satisfies

$$-\Delta\overline{G}_\kappa + \kappa^2\overline{G}_\kappa = 4\pi\delta_0 \qquad \text{in } \mathbb{R}^3. \tag{1.2}$$

It plays an important role in the theory of elementary particles, see [45]. For this energy $\mathcal{P}_\gamma$, comprising short-range interaction (surface tension) and attractive Yukawa potential, a stationary set Ω may separate into well ordered disjoint components. This gives rise to formation of patterns in the limit of several nonlocal reaction–diffusion models from physics, biology and chemistry. Indeed, systems with competing short range and Coulomb-type attractive interaction provides sharp interfaces to the Ohta–Kawasaki model of block

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polymers and the activator–inhibitor reaction–diffusion system, see for instance [18,29,43] and the references therein. The paper of Muratov [26] contains a rigorous derivation of (1.1) from the general mean-field free energy functional, with $\gamma \approx \kappa^2$ and $\gamma, \kappa \to 0$. Equilibrium patterns such as spots, stripes and the annuli, together with morphological instabilities have been also studied in [26]. At certain energy levels, (1.1) provides the sharp interface for the Ohta–Kawasaki model of diblock copolymer melt, see Muratov [27] and the work of Goldman, Muratov and Serfaty in [16,17] for related variant of Γ-convergence on the 2-dimensional torus.

In [28], Oshita showed that (1.1) is the Γ-limit for a class of activator–inhibitor reaction–diffusion system, see also Petrich and Goldstein in [30], for the two dimensional case.

The question of minimizing $\mathcal{P}_\gamma$ has been also raised by H. Knüpfer, C. Muratov and M. Novaga in [24], for physical and mathematical perspectives. They speculated that bifurcations from trivial to nontrivial solutions can be expected. Moreover they predict the "pearl-necklace morphology" (or string-of-pearl) exhibited by long polyelectrolyte molecules in poor solvents, [12,15]. In a geometric point of view, this is alike the cylinder-to-sphere transition, and known as Delauney surfaces (or unduloids). The unduloids are constant mean curvature surfaces of revolution interpolating between a straight cylinder to a string of tangent round spheres. Here, we will consider the limit configuration, and we prove existence of stationary sets for $\mathcal{P}_\gamma$ formed by a periodic string of nearly round balls.

It is known, see for instance [14,26], that a stationary set (or critical point) Ω for the functional $\mathcal{P}_\gamma$ satisfies the following Euler–Lagrange equation

$$H_{\partial\Omega}(x) + \gamma \int_\Omega G_\kappa(|x-y|)dy = Const. \qquad \text{for all } x \in \partial\Omega, \tag{1.3}$$

where $H_{\partial\Omega}$ is the mean curvature (positive for a ball) of $\partial\Omega$. Here and in the following, for every set Ω with C^2 boundary (not necessarily bounded), we define the function

$$\mathcal{H}_\Omega : \partial\Omega \to \mathbb{R}$$

given by

$$\mathcal{H}_\Omega(x) = H_{\partial\Omega}(x) + \gamma \int_\Omega G_\kappa(|x-y|)dy, \qquad \text{for every } x \in \partial\Omega. \tag{1.4}$$

In this paper, *equilibrium patterns* are sets $\Omega \subset \mathbb{R}^3$ for which $\mathcal{H}_\Omega \equiv Const$ on $\partial\Omega$.

Our aim is to construct nontrivial unbounded equilibrium patterns Ω. These sets are multiply periodic and bifurcate from lattices of round *spheres*, straight *cylinders* and *slabs*. In particular, we obtain stationary sets for $\mathcal{P}_\gamma$ made by lattices of near-spheres centered at any M-dimensional Bravais lattice. When $M = 1$, these are adjacent nearly-spherical sets centered at a straight line with equal gaps between them. The cylinders, result from the 2-dimensional existence of spherical patterns with the corresponding Yukawa potential given by a modified Bessel function. These configurations appear for all $\gamma \in (0, \gamma_N)$, for some positive constant γ_N. Finally, we obtain lamellar structures (modulated slabs) bifurcating from parallel planes. The bifurcations of these modulated slabs occur as long as $\frac{\kappa\sqrt{\kappa^2+1}}{\sqrt{\kappa^2+1}-\kappa} < 2\pi\gamma < (\kappa^2+1)^{3/2}$.

We first describe our doubly periodic slabs. We consider domains of the form,

$$\Omega_\varphi = \left\{(t,z) \in \mathbb{R}^2 \times \mathbb{R} \,:\, -\varphi(t) < z < \varphi(t)\right\} \subset \mathbb{R}^3, \tag{1.5}$$

where $\varphi : \mathbb{R}^2 \to (0,\infty)$ is $2\pi\mathbb{Z}^2$-periodic and satisfies some symmetry properties. Indeed, we let $C^{k,\gamma}(\mathbb{R}^2)$ denote the space of $C^k(\mathbb{R}^2)$ bounded functions u, with bounded derivatives up to order k and with $D^k u$ having finite Hölder seminorm of order $\alpha \in (0,1)$. The space $C^{k,\alpha}_p(\mathbb{R}^2)$ denotes the space of functions in $C^{k,\alpha}(\mathbb{R}^2)$ which are $2\pi\mathbb{Z}^2$-periodic. Finally, we define

$$C^{k,\alpha}_{p,\mathcal{R}} = \left\{\varphi \in C^{k,\alpha}_p(\mathbb{R}^2) \,:\; \varphi(t_1,t_2) = \varphi(t_2,t_1) = \varphi(-t_1,t_2), \text{ for all } (t_1,t_2) \in \mathbb{R}^2\right\}.$$

The spaces $C^{k,\alpha}_{p,\mathcal{R}}$ and $C^{k,\alpha}_p(\mathbb{R}^2)$ are equipped with the standard Hölder norm of $C^{k,\alpha}(\mathbb{R}^2)$.