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A note on the optimal boundary regularity for the planar generalized p-Poisson equation

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ABSTRACT

In this note, we establish sharp regularity for solutions to the following generalized p-Poisson equation

$$-\ div\ \left(\langle A\nabla u, \nabla u\rangle^{\frac{p-2}{2}} A\nabla u\right) = -\ div\ \mathbf{h} + f$$

in the plane (i.e. in $\mathbb{R}^2$) for $p > 2$ in the presence of Dirichlet as well as Neumann boundary conditions and with $\mathbf{h} \in C^{1-2/q}$, $f \in L^q$, $2 < q \leq \infty$. The regularity assumptions on the principal part A as well as that on the Dirichlet/Neumann conditions are exactly the same as in the linear case and therefore sharp (see Remark 2.8 below). Our main results Theorems 2.6 and 2.7 should be thought of as the boundary analogues of the sharp interior regularity result established in the recent interesting paper by Araujo et al. (2017) in the case of

$$-\ div\ (|\nabla u|^{p-2}\nabla u) = f \tag{1}$$

for more general variable coefficient operators and with an additional divergence term.

1. Introduction

In this paper, we study sharp $C^{1,\alpha}$ regularity estimates in the plane for

$$-\ div\ \left(\langle A\nabla u, \nabla u\rangle^{\frac{p-2}{2}} A\nabla u\right) = -\ div\ \mathbf{h} + f, \tag{2}$$

with Dirichlet/Neumann boundary conditions when $\mathbf{h} \in C^{1-2/q}$, $f \in L^q$, $2 < q \leq \infty$. In the linear case, i.e. when $p = 2$, it is well known that solutions to

$$-\ \Delta u = f \in L^\infty(B_1)$$

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are of class $C^{1,\alpha}_{loc}$ for every $\alpha < 1$ but need not be in $C^{1,1}$. In the degenerate setting $p > 2$, the situation is quite different and the smoothing effect of the operator is less prominent as the following radially symmetric example shows. More precisely for $(0 < a < 1)$

$$u(x) = |x|^{1+a} \tag{3}$$

(as mentioned in [2]) we have that

$$div\ (|\nabla u|^{p-2}\nabla u) = c_{a,p}|x|^{ap-a-1} \tag{4}$$

for some constant $c_{a,p} \neq 0$ if $a \neq 3/(p-1)$. The RHS is in $L^q(B_1)$ if $a > (1-2/q)/(p-1)$. This example shows that the best regularity that one can expect for solutions to (1) is $C^{1,(1-2/q)/(p-1)}$. In fact, this example gives rise to the following well known conjecture among the experts in this field and is referred to as the $C^{p'}$ conjecture.

Conjecture ($C^{p'}$ *Conjecture*). *Solutions to (1) are locally of class $C^{1,\frac{1}{p-1}}$ for $p > 2$ with $f \in L^\infty$.*

Note that the same example shows if $\mathbf{h} \in C^{1-2/q}$ then the best regularity we can expect for solutions to (2) is $C^{1,(1-2/q)/(p-1)}$.

Over here, we would like to mention that although the conjecture is open, nevertheless it is well known that solutions to (1) are locally of class $C^{1,\alpha}$ for some exponent α depending on p and n. See for instance, [5,10,16,17].

Very recently, the $C^{p'}$ conjecture has been solved in the planar case in [1]. The proof in [1] relies on a crucial global $C^{1,\alpha}$ estimate for p-harmonic functions in the planar case for some $\alpha > \frac{1}{p-1}$ combined with a certain geometric oscillation estimate which has its roots in the seminal paper of Caffarelli, see [4]. This very crucial global $C^{1,\alpha}$ estimate for planar p-harmonic functions follows from results in [3] which exploits the fact that the complex gradient of a p-harmonic function in the plane is a K-quasiregular mapping. Over here, we would like to mention that no analogous result concerning similar quantitative regularity for p-harmonic functions is known in higher dimensions.

In [14], (1) was studied with $f \in L^q$, $2 < q < \infty$ and optimal interior regularity was achieved in plane by Lindgren and Lindqvist. Recently in an interesting work of Araujo and Zhang, [2] more general p-Poisson equation (but $\mathbf{h} = 0$) is studied and some interior regularity is achieved. We assume $A \in C^{(1-2/q)/(p-1)}$ to achieve the same regularity as in the case of the p-Poisson equation.

In this paper, we make the observation that the ideas in [1] can be applied to more general variable coefficient equations with Dirichlet and Neumann boundary conditions. Over here, we would like to mention that although our work has been strongly motivated by that in [1], it has nonetheless required certain delicate adaptations in our setting due to the presence of the boundary datum. To apply certain iteration, as in [1], one needs to ensure smallness of boundary datum at each step of iteration, as the reader can see in the proofs of Theorems 3.4, 3.10. Moreover, we finally needed to combine the interior estimate and the estimate at the boundary in order to get a uniform estimate and this required a bit of subtle analysis as well, as can be seen in the proof of Theorem 2.6 after Theorem 3.6. In closing, we would like to mention two other interesting results which are closely related to this article. In [8] (see also [9]), Kuusi and Mingione established the continuity of ∇u assuming f in the Lorentz space $L(n, \frac{1}{p-1})$ and where the principal part is sightly more general as in [2] and has Dini dependence in x. Moreover, a moduli of continuity of ∇u is also established in the same article when the principal part has Hölder dependence in x and $f \in L^q$ for $n < q \leq \infty$.

The paper is organized as follows: In Section 2, in order to keep our paper self contained, we gather some known regularity results and then state our main results. In Section 3, we prove our main result by following