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Existence of a weak solution to the p-Sobolev flow

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ABSTRACT

In this paper, we study a doubly nonlinear parabolic equation, called the p-Sobolev flow here, which is the classical Yamabe flow on a bounded domain in Euclidean space in the case $p = 2$. We show the existence of a weak solution to the p-Sobolev flow without geometric assumptions.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$ and let $T > 0$ be arbitrarily given and fixed. Then we consider the following doubly nonlinear parabolic initial–boundary problem (1.1):

$$\begin{cases} \partial_t(|u|^{q-1}u) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{in} \quad \Omega \times (0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(0) = u_0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $2 \leq p < n$, $q := \frac{np}{n-p} - 1$, and $u = (u^i) = (u^i(x,t)), i = 1, \dots, k$, is a vector valued function, defined for $(x,t) \in \Omega \times [0,T]$ with values into $\mathbb{R}^k$. We call (1.1) as *p-Laplace flow* and *Yamabe flow*, respectively if $p > 2$ and $p = 2$, and collectively refer (1.1) as *p-Sobolev flow*. Our main purpose is to prove the following Theorem 1.1 related to the existence of a weak solution to (1.1).

Theorem 1.1. *Assume that the $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, for any $T > 0$ there exists a weak solution to (1.1).*

The Yamabe flow is originally introduced by Hamilton in his study of the so-called Yamabe problem, the existence of a conformal metric of constant curvature on $n(\geq 3)$-dimensional closed Riemannian manifolds [5].

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Let (M, g_0) be a $n(\geq 3)$-dimensional smooth, closed Riemannian manifold with scalar curvature $R_0 = R_{g_0}$. The Yamabe flow is given by the heat flow equation

$$u_t = (s - R)u = u^{-\frac{4}{n-2}}(c_n \Delta_{g_0} u - R_0 u) + su, \tag{$*$}$$

where $u = u(t), t \geq 0$ is a positive function on M such that $g(t) = u(t)^{\frac{4}{n-2}} g_0$ is a conformal change of a Riemannian metric g_0, with volume $\mathrm{Vol}(M) = \int_M dvol_g = \int_M u^{\frac{2n}{n-2}} dvol_{g_0} = 1$, having total curvature

$$s := \int_M (c_n |\nabla u|^2_{g_0} + R_0 u^2)\, dvol_{g_0} = \int_M R\, dvol, \quad c_n := \frac{4(n-1)}{n-2}.$$

Hamilton [5] proved some convergence of the Yamabe flow as $t \to \infty$ in some geometric conditions. Under the assumption that (M, g_0) is positive scalar curvature and locally conformal flat, Ye [7] showed the global existence of the Yamabe flow and its convergence as $t \to \infty$ to a metric of constant scalar curvature. Schwetlick–Struwe [6] succeeded in obtaining the asymptotic convergence of the Yamabe flow in the case $3 \leq n \leq 5$, under an appropriate condition of $Y(M, g_0)$ for an initial positive scalar curvature. On the other hand, in a simpler geometrical case when (M, g_0) is a bounded domain $(\Omega, g_{\mathbb{R}^n})$ in $\mathbb{R}^n$, the Yamabe flow is our p-Sobolev flow (1.1) with $p = 2$, but the curvature conditions above are not verified. In this paper, we prove the existence of a weak solution of (1.1). Here we shall recall the fundamental result due to Alt-Luckhaus [1], who addressed the following initial–boundary problem for a monotone vector field $\boldsymbol{b}(u)$ and continuous functions $a(u, \boldsymbol{b}(u)), f(\boldsymbol{b}(u))$ with polynomial growth conditions:

$$\begin{cases} \partial_t \boldsymbol{b}(u) = \mathrm{div}\,(a(u, \boldsymbol{b}(u))) + f(\boldsymbol{b}(u)) & \text{in} \quad \Omega \times (0,T) \\ u = u_D \quad \text{in} \quad \partial\Omega \times (0,T) \\ u(0) = u_0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

where, $\boldsymbol{b}(u)$ $(u \in \mathbb{R}^k)$ is *monotone* if $(\boldsymbol{b}(v) - \boldsymbol{b}(w)) \cdot (v - w) \geq 0$ holds for any $v, w \in \mathbb{R}^k$. For (1.1), $\boldsymbol{b}(u) := |u|^{q-1}u$ $(q > 1)$, which is actually monotone (Lemma A.1). In [1] replaced the time derivative term in (1.2) by the backward difference quotient on time (1.2) is reduced into elliptic problem, which can be solved by Galerkin's produce and Minty's monotone trick. This basic procedure is also used in this paper.

2. Preliminaries

For a vector-valued function $v : \Omega \to \mathbb{R}^k$, we use some function spaces. For each p, $1 \leq p < \infty$, let $L^p(\Omega)$ denote the Banach spaces of vector-valued measurable functions $v : \Omega \to \mathbb{R}^k$ that are pth integrable on Ω, with the norm

$$\|v\|_{L^p(\Omega)} := \left(\int_\Omega |v(x)|^p\, dx\right)^{1/p}$$

and for $p = \infty$, $L^\infty(\Omega)$ is the Banach space of essentially bounded vector-valued function with the norm

$$\|v\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |v(x)|.$$

For $1 \leq p < \infty$, the Sobolev space $W^{1,p}(\Omega)$ is the Banach space of vector-valued functions that are weakly differentiable and their weak derivatives are pth integrable on Ω, with the norm

$$\|v\|_{W^{1,p}(\Omega)} := \left(\int_\Omega |v|^p + |\nabla v|^p\, dx\right)^{1/p},$$

where $\nabla v = (\nabla v^1, \ldots, \nabla v^k)$ denotes the gradient of $v = (v^1, \ldots, v^k)$ in a distribution sense, and let $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}}$. For $1 \leq p, q \leq \infty$, $L^q(0,T;\, L^p(\Omega))$ is a function