Nonlinear Analysis 175 (2018) 173–190

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Green function and Martin kernel for higher-order fractional Laplacians in balls

Nicola Abatangelo[a,*], Sven Jarohs[b], Alberto Saldaña[c]

[a] *Département de Mathématique, Université Libre de Bruxelles CP 214, boulevard du Triomphe, 1050 Ixelles, Belgium*
[b] *Institut für Mathematik, Goethe-Universität, Robert-Mayer-Straße 10, 60054 Frankfurt, Germany*
[c] *Institut für Analysis, Karlsruher Institut für Technologie, Englerstraße 2, 76131 Karlsruhe, Germany*

ARTICLE INFO

Article history:
Received 26 March 2018
Accepted 31 May 2018
Communicated by Enzo Mitidieri

MSC:
35C15
35C04
35S15
35B50

Keywords:
Boggio's formula
s-harmonic functions
Maximum principles

ABSTRACT

We give the explicit formulas for the Green function and the Martin kernel for all integer and fractional powers of the Laplacian $s > 1$ in balls. As consequences, we deduce interior and boundary regularity estimates for solutions to linear problems and positivity preserving properties. Our proofs rely on a characterization of suitable s-harmonic functions and on a differential recurrence equation.

1. Introduction

In this paper, we show that the Green function $\mathscr{G}_s$ and the Martin kernel M_s for any power $s > 1$ of the Laplacian $(-\Delta)^s$ in the unit ball $B \subset \mathbb{R}^N$, $N \in \mathbb{N}$, are given by

$$\mathscr{G}_s(x,y) := k_{N,s}|x-y|^{2s-N} \int_0^{\rho(x,y)} \frac{v^{s-1}}{(v+1)^{\frac{N}{2}}}\,dv \qquad \text{for } x,y \in \mathbb{R}^N,\ x \neq y, \tag{1.1}$$

where

$$\rho(x,y) := \frac{(1-|x|^2)_+(1-|y|^2)_+}{|x-y|^2}, \qquad k_{N,s} := \frac{\Gamma(\frac{N}{2})}{\pi^{\frac{N}{2}} 4^s \Gamma(s)^2} \tag{1.2}$$

* Corresponding author.
E-mail addresses: nicola.abatangelo@ulb.ac.be (N. Abatangelo), jarohs@math.uni-frankfurt.de (S. Jarohs), alberto.saldana@partner.kit.edu (A. Saldaña).

https://doi.org/10.1016/j.na.2018.05.019
0362-546X/

and

$$M_s(x,\theta) \;=\; \lim_{B\ni y\to\theta}\frac{\mathscr{G}_s(x,y)}{(1-|y|^2)^s} = \frac{k_{N,s}}{s}\,\frac{(1-|x|^2)_+^s}{|\theta-x|^N} \qquad \text{for } x\in\mathbb{R}^N,\ \theta\in\partial B. \tag{1.3}$$

For $m\in\mathbb{N}_0$, $\sigma\in(0,1)$, and $s=m+\sigma$, the operator $(-\Delta)^s$ can be defined pointwisely via finite differences (see [4, equation (1)]), namely, for $u\in C^{2s+\alpha}(U)\cap L^\infty(\mathbb{R}^N)$,

$$\begin{aligned}
&(-\Delta)^s u(x) := \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\frac{\delta_{m+1}u(x,y)}{|y|^{N+2s}}\,dy, \qquad x\in\mathbb{R}^N,\\
\text{where}\qquad &\delta_{m+1}u(x,y) := \sum_{k=-m-1}^{m+1}(-1)^k\binom{2(m+1)}{m+1-k}u(x+ky) \qquad \text{for } x,y\in\mathbb{R}^N
\end{aligned} \tag{1.4}$$

is a finite difference of order $2(m+1)$, and $c_{N,s}$ is a *positive* normalization constant (for the precise value, see [4, equation (2)]) such that the Fourier symbol of $(-\Delta)^s$ is $|\xi|^{2s}$ (see [22, Lemma 25.3] or [4, Theorem 1.8]); moreover, if $u\in C^{2s+\alpha}(U)\cap L^\infty(\mathbb{R}^N)$ then $(-\Delta)^s u(x) = (-\Delta)^m(-\Delta)^\sigma u(x)$ for every $x\in U$ (see [4, Corollary 1]), but in general the fractional Laplacian $(-\Delta)^\sigma$ cannot be interchanged freely with the Laplacian $(-\Delta)$, this would require extra regularity assumptions on u, particularly across the boundary ∂U (see [4]), but for $u\in C_c^\infty(\mathbb{R}^N)$ we have

$$(-\Delta)^s u = (-\Delta)^\sigma(-\Delta)^m u = (-\Delta)^m(-\Delta)^\sigma u \quad \text{in } \mathbb{R}^N.$$

For the relevance and applications of the higher-order fractional Laplacian we refer to [3,21].

Our main result regarding the Green function is the following.

Theorem 1.1. *Let $s>0$, $N\in\mathbb{N}$, $f\in C^\alpha(\overline{B})$ for some $\alpha\in(0,1)$, and*

$$u:\mathbb{R}^N\to\mathbb{R} \quad \textit{be given by} \quad u(x) := \int_B \mathscr{G}_s(x,y)\,f(y)\,dy. \tag{1.5}$$

Then $u\in C^{2s+\alpha}(B)\cap C_0^s(B)$ is the unique pointwise solution (in $\mathscr{H}_0^s(B)$) of

$$(-\Delta)^s u = f \quad \textit{in } B, \qquad u\equiv 0 \quad \textit{on } \mathbb{R}^N\setminus B, \tag{1.6}$$

and there is $C>0$ such that

$$\|\operatorname{dist}(\cdot,\partial B)^{-s}u\|_{L^\infty(B)} < C\|f\|_{L^\infty(B)}. \tag{1.7}$$

Theorem 1.1 was known for $s\in\mathbb{N}$ [7] and for $s\in(0,1)$ [5,8]. While preparing the last version of this work we learned about a preprint of [10], where the authors show independently the validity of Boggio's formula for all $s\in(1,\infty)\setminus\mathbb{N}$ considering only smooth functions with compact support as right-hand sides. Our proofs are very different, the approach in [10] relies on covariance under Möbius transformations and computations using Hypergeometric functions, whereas our proofs are based on an induction argument detailed below.

The function $\mathscr{G}_s$ is known as *Boggio's formula*, see [7,8,10,14]. Since $\mathscr{G}_s$ is a positive function, Theorem 1.1 shows that problems on balls enjoy a positivity preserving property. This is not the case for general domains, see [3]. Our proof of Theorem 1.1 is based on a differential recurrence formula for $\mathscr{G}_s$ in terms of $\mathscr{G}_{s-1}$ and an explicit function P_{s-1} which is $(s-1)$-harmonic in the ball, see Lemma 3.1. Since the validity of Boggio's formula is known for $s\in(0,1]$ (in view, for example, of the direct computations in [8]), this allows us to implement an induction argument to extend this result to all $s>1$. We remark that our approach also provides an alternative proof for $s\in\mathbb{N}$. Two key elements in the proof are an elementary pointwise calculation of $-\Delta_x\mathscr{G}_s(x,y)$ for $y\neq x$ and $s>1$ (see Lemma 3.1) and the introduction of *higher-order Martin kernels* (1.3), which we use to characterize a large class of s-harmonic functions, see Theorem 1.2. Martin