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On the periodic orbits, shadowing and strong transitivity of continuous flows

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We prove that chaotic flows (i.e. flows that satisfy the shadowing property and have a dense subset of periodic orbits) satisfy a reparametrized gluing orbit property similar to the one introduced in Bomfim and Varandas (2015). In particular, these are strongly transitive in balls of uniform radius. We also prove that the shadowing property for a flow and a generic time-t map, and having a dense subset of periodic orbits hold for a C^0-Baire generic subset of Lipschitz vector fields, that generate continuous flows. Similar results also hold for C^0-generic homeomorphisms and, in particular, we deduce that chain recurrent classes of C^0-generic homeomorphisms have the gluing orbit property.

1. Introduction

We will say that a continuous flow is *chaotic* if it has a dense set of periodic orbits and it satisfies the shadowing property. The pursuit of periodic trajectories for flows associated with systems of differential equations has been a central issue in dynamics since the work of Poincaré on celestial mechanics. Knowing the abundance of periodic orbits on the state phase gives us information about the dynamical complexity of the system. In rough terms the shadowing property, which consists of a reconstruction of a true orbit for the dynamics provided a set of points that form approximately an orbit, appears in many branches of dynamical systems and is also often related with how complex the system is (see [12,16]). Actually, the computational estimates, fitted with a certain error of orbits, are meaningless if they cannot be realized by genuine orbits of the given system.

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In this paper we prove that continuous flows with a dense set of periodic orbits and the shadowing property satisfy a reparametrized gluing orbit property (see Section 2 for definition). This property says that any finite number of compact pieces of orbits of the flow can be shadowed by a true reparametrized orbit. The need of the reparametrizations is natural in the context of flows, in order to compensate the sliding effect of close orbits along the flow direction (see e.g. [16]). Moreover, this property implies that such flows are strongly transitive: *the shortest hitting time from a ball to any other ball of the same radius is uniformly bounded above by a constant depending only on the radius.*

In this time-continuous setting one cannot expect to deduce all the flows to be topologically mixing. Indeed, any Anosov flow obtained as time-one suspension flow of an Anosov diffeomorphism is chaotic but is clearly not topologically mixing. Since the minimal requirement for a vector field to generate a flow (by uniqueness of solution of an ordinary differential equation) is the Lipschitz regularity of the vector field, we will consider throughout $\mathfrak{X}^{0,1}(M)$ as the space of Lipschitz continuous vector fields on a compact Riemannian manifold M endowed with the C^0-topology.

A main goal here is to study the transitivity of C^0 flows generated by Lipschitz vector fields and to establish a weaker counterpart of Oxtoby–Ulam theorem: C^0-generic flows are topologically transitive. Recall that Oxtoby and Ulam proved that generic volume-preserving homeomorphisms are ergodic (see e.g. [1]).

In our approach, the C^0 genericity of transitivity arises as a byproduct of a stronger characterization for typical C^0 flows generated by Lipschitz vector fields. First we prove that both the denseness of periodic orbits and the shadowing property are C^0-generic on vector fields that generate continuous flows (cf. Theorems 2 and 4), extending previous results by Coven, Maden, Nitecki [9] and Kościelniak and Odani [13,15] to the continuous-time setting. Quite surprisingly, these two properties altogether are enough to assure that the chain recurrent classes of these flows satisfy a reparametrized gluing orbit property and, consequently, are strongly transitive (cf. Theorem 1). The gluing orbit property is a weakening of the notion of specification and roughly means that any finite pieces of orbits can be shadowed by a true orbit where the time lag between the pieces of orbits is bounded above by a constant that depends only on the shadowing distance (see Definition 2.3 for a precise statement). The later bridges uniformly hyperbolic and non-hyperbolic dynamics (e.g. it holds for irrational translation flows on the torus) and, despite the fact that it does not imply the flow to be topologically mixing it assures some positive frequency of visits to balls (cf. [6] for more examples and other properties). This is an interesting fact on its own, as the gluing orbit property constitutes a useful tool to describe the thermodynamic formalism, large deviations, recurrence and multifractal analysis (see [6–8,17] and references therein). The Oxtoby–Ulam conjecture cannot be extended to the C^1-topology, where the existence of sinks/sources constitutes a non-removable obstruction to ergodicity. In the same way, by [6, Theorem B] there exists a C^1-residual subset of vector fields $\mathfrak{X}^1(M)$ such that whenever the flow generated by it has the gluing orbit property it is indeed a transitive Anosov flow. In consequence, if M is a compact manifold that does not admit Anosov flows then there exists a C^1-residual subset of vector fields that do not have the gluing orbit property.

A second main goal here is to study different shadowing properties for flows and for their time-t maps. In general the time-t map of a flow is harder to describe than the flow itself. A simple illustration is given by C^1-smooth uniformly hyperbolic flows, in which case their time-t maps are just partially hyperbolic. In particular, it is most common that topological and ergodic properties for the time-t maps do not follow immediately those of their analogs for the flow. Taking this into account, here we are mainly interested in describing topological features of the homeomorphisms that arise as time-t maps of typical continuous flows. Historically, there are several notions of shadowing for flows which have been introduced in order to describe the strong properties of reconstruction of orbits exhibited by hyperbolic and non-uniformly hyperbolic flows.

Here, as a result of independent interest we focus on the discrete-time dynamics and also we prove that generic time-t maps of C^0-generic vector fields are homeomorphisms that satisfy the classical shadowing property (see Theorem 3 for the precise statement).