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Rigidity of Einstein metrics as critical points of quadratic curvature functionals on closed manifolds

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ABSTRACT

In this paper, we prove some rigidity results for the Einstein metrics as the critical points of a family of known quadratic curvature functionals on closed manifolds, characterized by some point-wise inequalities. Moreover, we also provide a few rigidity results that involve the Weyl curvature, the trace-less Ricci curvature and the Yamabe invariant, accordingly.

1. Introduction

In this paper, we always assume that M^n is a closed manifold of dimension $n \geq 3$ and g a Riemannian metric on M^n with the Riemannian curvature tensor R_{ijkl}, the Ricci tensor R_{ij} and the scalar curvature R. It is well-known that any Einstein metric g must be critical for the Einstein–Hilbert functional

$$\mathcal{H} = \int_M R$$

defined on the space $\mathscr{M}_1(M^n)$ of equivalence classes of smooth Riemannian metrics of volume one on M^n. On the other hand, Catino considered in [4] the following family of quadratic curvature functionals

$$\mathcal{F}_t = \int_M |R_{ij}|^2 + t \int_M R^2, \quad t \in \mathbb{R} \tag{1.1}$$

which are also defined on $\mathscr{M}_1(M^n)$, and proved some related rigidity results. Furthermore, it has been observed in [2] that every Einstein metric is a critical point of $\mathcal{F}_t$ for all $t \in \mathbb{R}$, see (2.5). But the converse of this conclusion is not true in general.

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Therefore it is natural to ask that under what conditions a critical metric for the functionals $\mathcal{F}_t$ must be a Einstein one. In fact, there have been a number of interesting conclusions to this problem, for example, under some suitable curvature conditions [4,14,15], or under some integral conditions [9,13]. For other development in this direction, we refer the readers to [1,3,10,11] and the references therein.

Recall that the Yamabe invariant $Y_M([g])$ is defined by

$$\begin{aligned} Y_M([g]) &= \inf_{\tilde{g}\in[g]} \frac{\int_M \tilde{R}\, dv_{\tilde{g}}}{(\int_M dv_{\tilde{g}})^{\frac{n-2}{n}}} \\ &= \frac{4(n-1)}{n-2} \inf_{u\in W^{1,2}(M^n)} \frac{\int_M |\nabla u|^2\, dv_g + \frac{n-2}{4(n-1)}\int_M Ru^2\, dv_g}{(\int_M |u|^{\frac{2n}{n-2}}\, dv_g)^{\frac{n-2}{n}}}, \end{aligned} \tag{1.2}$$

where $[g]$ is the conformal class of the metric g. It then follows that

$$\begin{aligned} &\frac{n-2}{4(n-1)} Y_M([g]) \Big(\int_M |u|^{\frac{2n}{n-2}}\, dv_g\Big)^{\frac{n-2}{n}} \\ &\quad \le \int_M |\nabla u|^2\, dv_g + \frac{n-2}{4(n-1)}\int_M Ru^2\, dv_g, \end{aligned} \tag{1.3}$$

for all $u \in W^{1,2}(M^n)$. Moreover, $Y_M([g])$ is positive if and only if there exists a conformal metric in $[g]$ with everywhere positive scalar curvature.

In the present paper, by using some pinching conditions involving the Weyl curvature, the trace-less Ricci curvature and the Yamabe invariant, we aim to prove a number of rigidity theorems for the Einstein metrics considered as the critical points of the functional family $\mathcal{F}_t$ $(t \in R)$. For convenience, we shall use $\mathring{\mathrm{Ric}}$ and W throughout this paper to denote the trace-less Ricci curvature and the Weyl curvature, respectively.

Our main results are stated as follows.

Theorem 1.1. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \ge 3$ with positive scalar curvature and g is a critical metric for the functional family $\mathcal{F}_t$ over $\mathscr{M}_1(M^n)$, where*

$$\begin{cases} t < -\dfrac{5}{12}, & \text{if } n = 3; \\ t < -\dfrac{1}{3}, & \text{if } n = 4; \\ t \le -\dfrac{n}{4(n-1)}, & \text{if } n \ge 5. \end{cases} \tag{1.4}$$

Suppose that

$$\begin{aligned} &\Big|W - \frac{n-4}{\sqrt{2n}(n-2)} \mathring{\mathrm{Ric}} \owedge g\Big| \\ &\quad < -\sqrt{\frac{2}{(n-1)(n-2)}} \left(\frac{2(n-2)+2n(n-1)t}{n} + 1\right) R. \end{aligned} \tag{1.5}$$

Then (M^n, g) must be of Einstein.

In particular, when $n = 3$, we have $W = 0$ automatically. On the other hand, from (2.2) it is seen that an Einstein manifold M^3 with positive scalar curvature must be of constant positive sectional curvature. Moreover, it follows from Lemma 2.3 that $|\mathring{\mathrm{Ric}} \owedge g| = 2|\mathring{\mathrm{Ric}}|$ on M^3. Consequently, the following conclusion is immediate by Theorem 1.1.