Nonlinear Analysis 174 (2018) 1–16

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Radius of analyticity for the Camassa–Holm equation on the line

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ARTICLE INFO

Article history:
Received 20 November 2017
Accepted 9 April 2018
Communicated by Enzo Mitidieri

MSC:
primary 35Q53
37K10
35C07

Keywords:
Camassa–Holm equation
Cauchy problem
Analytic data
Global in time solutions
Analytic spaces
Exponential lower bound on the radius of analyticity

ABSTRACT

Using estimates in Sobolev spaces we prove that the solution to the Cauchy problem for the Camassa–Holm equation on the line with analytic initial data $u_0(x)$ and satisfying the McKean condition, that is the quantity $m_0(x) = (1-\partial_x^2)u_0(x)$ does not change sign, is analytic in the spatial variable for all time. Furthermore, we obtain explicit lower bounds for the radius of spatial analyticity $r(t)$ given by $r(t) \geq A^{-1}(1+C_1Bt)^{-1}\exp\{-C_0\|u_0\|_{H^1}t\}$, where A, B, C_1 andC_0 are suitable positive constants.

1. Introduction and main result

We consider the Cauchy problem for the Camassa–Holm (CH) equation on the line

$$\begin{cases} u_t = -u\partial_x u - \partial_x(1-\partial_x^2)^{-1}\left[u^2 + \dfrac{1}{2}(\partial_x u)^2\right] \\ u(x,0) = u_0(x), \quad x \in \mathbb{R},\ t \geq 0, \end{cases} \tag{1.1}$$

and study the problem of analyticity of the smooth solutions for initial data $u_0(x)$ that are analytic on the line and can be extended as holomorphic functions in a strip around the x-axis. Under the condition that the McKean quantity $(1-\partial_x^2)u_0(x)$ (see [38,39]) does not change sign we obtain explicit lower bounds on the radius of spatial analyticity $r(t)$ at any time $t \geq 0$, which is given by $(1+t)^{-1}\exp\{-Lt\}$, where L is a positive constant, that will be described more precisely later. We recall that the Cauchy problem for the CH equation is globally well-posed for initial data in $H^\infty(\mathbb{R}) = \bigcap_{s\geq 0} H^s(\mathbb{R})$ and satisfying the McKean condition

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https://doi.org/10.1016/j.na.2018.04.007
0362-546X/

(see, e.g., [43,3]). Furthermore, under the above analyticity assumption on initial data, it has been shown in [3] that the solution to the CH Cauchy problem is globally analytic in $x \in \mathbb{R}$, $t \geq 0$, and a lower bound of double exponential decay was derived for the radius of space analyticity at later times. More precisely, the lower bound for $r(t)$ is of the form $L_3 \exp(-L_1 \exp(L_2 t))$, where L_1, L_2 and L_3 are appropriate positive constants. Therefore, this work improves the double exponential decay of $r(t)$ to a single exponential decay.

The CH equation is an important integrable equation, which has been studied extensively for more than two decades now. It arose initially in the context of hereditary symmetries studied by Fuchssteiner and Fokas [19]. However, it was written explicitly as a water wave equation by Camassa and Holm [7], who derived it from the Euler equations of hydrodynamics using asymptotic expansions (see Constantin and Lannes [12] for a detailed discussion about the hydrodynamical relevance of CH and related equations). Also, they derived its peakon solutions (the simplest being $u(x,t) = ce^{-|x-ct|}$) and provided its Lax pair. In fact, CH is one of the two integrable members of the b-family of equations

$$u_t + uu_x + \partial_x(1-\partial_x^2)^{-1}\Big[\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2\Big] = 0, \tag{1.2}$$

(the other being the Degasperis–Procesi (DP) equation [18]) which has peakon and multipeakon solutions for all values of the real parameter b [27,17,40]. We note that CH corresponds to $b = 2$ and DP corresponds to $b = 3$. Peakon and more general traveling waves solutions for CH type equations have been studied in [4,8,14,15,28,30,24,34,37]. The CH equation is well-posed in Sobolev spaces H^s for $s > 3/2$ and ill-posed for $s < 3/2$. Local well-posedness and continuity properties of the solution map are proved in [13,21,23,35,43,16,20]. Furthermore, global well-posedness and blow up in finite time results are proved in [6,9,10,39]. Finally, we note that the literature about the Camassa–Holm and related equations is extensive. For more results about them we refer the reader to [11,12,25,26,36,29,41,46] and the references therein.

Next, we introduce the spaces needed for stating our main result precisely. First, for $\theta \geq 0$ and $\delta > 0$, we define the following Hilbert space of analytic functions on the real line

$$G^{\delta,\theta}(\mathbb{R}) = \{\varphi \in L^2(\mathbb{R};\mathbb{R}) : \|\varphi\|^2_{G^{\delta,\theta}} \doteq \int_{\mathbb{R}} \langle\xi\rangle^{2\theta} e^{2\delta|\xi|} |\widehat{\varphi}(\xi)|^2 d\xi < \infty\}, \tag{1.3}$$

where $\langle\xi\rangle \doteq (1+\xi^2)^{1/2}$ is a standard notation in the literature of Sobolev spaces. We point out that the elements of $G^{\delta,\theta}(\mathbb{R})$, $\theta > \frac{1}{2}$, have an analytic extension on a strip of the complex plane around the x-axis with width 2δ (see [2]). Also, we shall need the following spaces of analytic functions, which were first introduced by Kato and Masuda (see [31]). For each $r > 0$, we define $A(r)$ to be the set of all real-valued functions f that can be extended analytically in the strip $S(r)$ of width $2r$ around the x-axis in the complex plane and also belong in $L^2(S(r'))$ for every $0 < r' < r$. More precisely, we have

$$A(r) \doteq \{f : \mathbb{R} \to \mathbb{R} : f(z) \text{ is analytic in } S(r)\} \cap \{f : f \in L^2(S(r')) \text{ for all } 0 < r' < r\}, \tag{1.4}$$

where $S(r) = \{z \in \mathbb{C} : -\infty < \operatorname{Re} z < \infty, -r < \operatorname{Im} z < r\}$. Also, we note that $A(r)$ is a Fréchet space with these $L^2(S(r'))$-norms as the generating system of seminorms.

We would like to point out that the topology that we are going to use on $A(r)$ was also set by Kato and Masuda in [31]. More precisely, in [31] they use the following set of norms

$$\|f\|^2_{\sigma,s} = \sum_{j=0}^{\infty} \frac{1}{j!^2} e^{2j\sigma} \|\partial_x^j f\|_s^2, \qquad 0 < e^\sigma < r, \;\; s \geq 0, \tag{1.5}$$

where $\|\cdot\|_s$ denotes the standard Sobolev norm defined in terms of Fourier transform, that is for a test function $\varphi(x)$ we have

$$\|\varphi\|_s^2 = \int_{\mathbb{R}} (1+|\xi|^2)^s |\widehat{\varphi}(\xi)|^2 d\xi, \quad \text{with} \quad \widehat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \varphi(x)\, dx. \tag{1.6}$$