Nonlinear Analysis 174 (2018) 17–53

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

The Fredholm alternative for the p-Laplacian in exterior domains

Pavel Drábek[a], Ky Ho[b,*], Abhishek Sarkar[c]

[a] *Department of Mathematics, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic*
[b] *Department of Mathematics, Faculty of Sciences, Nong Lam University, Linh Trung Ward, Thu Duc District, Ho Chi Minh City, Vietnam*
[c] *NTIS, University of West Bohemia, Technická 8, 306 14 Plzeň, Czech Republic*

A R T I C L E I N F O

Article history:
Received 27 February 2017
Accepted 10 April 2018
Communicated by S. Carl

MSC:
35J92
35J60
35J20
35P30
35J62
35B40

Keywords:
p-Laplacian
Fredholm alternative
The first eigenvalue
Exterior domain
Variational method

A B S T R A C T

We investigate the Fredholm alternative for the p-Laplacian in an exterior domain which is the complement of the closed unit ball in $\mathbb{R}^N$ ($N \geq 2$). By employing techniques of Calculus of Variations we obtain the multiplicity of solutions. The striking difference between our case and the entire space case is also discussed.

1. Introduction

The Fredholm alternative for the p-Laplacian has been studied on both bounded domains in $\mathbb{R}^N$ and the entire space $\mathbb{R}^N$. In this paper we investigate the existence and multiplicity of solutions of the following problem

$$\begin{cases} -\Delta_p u = \lambda K(x)|u|^{p-2}u + h & \text{in } B_1^c, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian with $p > 1$, B_1^c is the complement of the closed unit ball B_1 in $\mathbb{R}^N$, $\lambda > 0$ is a parameter, the weight K and the function h will be specified later.

* Corresponding author.
E-mail addresses: pdrabek@kma.zcu.cz (P. Drábek), ngockyh@ntis.zcu.cz (K. Ho), sarkara@ntis.zcu.cz (A. Sarkar).

https://doi.org/10.1016/j.na.2018.04.006

In a bounded domain Ω of $\mathbb{R}^N$, similar problems (with $K(x) \equiv 1$) have been studied in numerous papers. For the references we refer the reader to survey papers by Takáč [17,18] and the references therein.

In the case of the entire space $\mathbb{R}^N$, Alziary et al. [1] studied the solvability of the equation

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u + f \quad \text{in } \mathbb{R}^N,\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \tag{1.2}$$

where $1 < p < N$ and the Sobolev space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is defined to be the completion of $C_c^1(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x\right)^{1/p}.$$

They studied problem (1.2) with a radially symmetric and measurable weight $m(x) = m(|x|)$ satisfying

$$0 < m(r) \le \frac{C}{(1+r)^{p+\mu}} \quad \text{a.e. in } [0,\infty), \tag{1.3}$$

with some constants $\mu > 0$ and $C > 0$. Let $\widetilde{\lambda}_1 > 0$ be the first eigenvalue and $\widetilde{\varphi}_1$ be the corresponding positive eigenfunction of $-\Delta_p$ in $\mathbb{R}^N$ relative to the weight $m(|x|)$; for the existence of the first eigenpair see, for example [14,15] and the references therein. For a given $f^* \in [\mathcal{D}^{1,p}(\mathbb{R}^N)]^*$ (the dual space of $\mathcal{D}^{1,p}(\mathbb{R}^N)$), satisfying $\langle f^*, \widetilde{\varphi}_1\rangle = 0$ (where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $[\mathcal{D}^{1,p}(\mathbb{R}^N)]^*$), the authors of [1] obtained the existence of at least one solution of (1.2) for $2 \le p < N$ with $\lambda = \widetilde{\lambda}_1$ and $f = f^*$, and for $1 < p < 2 \le N$ with $\lambda \in (\widetilde{\lambda}_1 - \epsilon, \widetilde{\lambda}_1 + \epsilon)$, $\epsilon > 0$ small, and f in a neighbourhood of f^*.

To obtain the existence of solutions, the authors of [1] used variational arguments but treated the two cases $1 < p < 2 \le N$ and $2 \le p < N$ in a different way. As a by-product, for the resonant case $\lambda = \widetilde{\lambda}_1$, they proved "a saddle point geometry" of the energy functional associated with (1.2) when $1 < p < 2 \le N$. On the other hand, they used an improved Poincaré inequality when $p \le 2 < N$ and showed that the energy functional has a "global minimizer geometry".

In the case of an exterior domain, Anoop et al. [2] discussed the existence of solution of problem (1.1) with a weaker assumption on weight than in [1] (see Definition 2.1 in the next section). By using the Fredholm alternative for the p-Laplacian due to Fučík et al. [11, Chapter II, Theorem 3.2], they obtained the existence of solutions for problem (1.1) when $\lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\}$ for some $\delta > 0$, where λ_1 is the first eigenvalue of $-\Delta_p$ in B_1^c relative to the weight K (see [2, Proposition 3.1]).

The goal of this paper is to obtain multiple solutions of (1.1) for the resonant case $\lambda = \lambda_1$ with a weaker assumption on the weight than in [1]. This work can be seen as a complement to the Fredholm alternative for the p-Laplacian in an exterior domain for the resonant case. It is worth mentioning that to deal with the resonant case, we apply the second order Taylor formula for the energy functional associated with (1.1) at the first eigenfunction φ_1 of $-\Delta_p$ in B_1^c. To apply Taylor formula, we need to employ weighted spaces in terms of φ_1 with the weights singular or degenerate, on the set $\{\nabla\varphi_1 = 0\}$. Surprisingly, the case of an exterior domain differs substantially from the case of the entire space $\mathbb{R}^N$. The important point to note here is the fact that, if K is radially symmetric and satisfies certain decay condition, the set $\{\nabla\varphi_1 = 0\}$ is a removable set (i.e., the set of zero capacity) in the case of the entire space $\mathbb{R}^N$, whereas this is not true in the case of an exterior domain (see, e.g., Remarks 2.14, 3.3 and 4.5). For this reason, to obtain a saddle point geometry of the energy functional in the resonant case when $1 < p < 2$, we need to introduce a new condition for the source term h, which is of independent interest.

In order to state our main results rigorously, we need to introduce several definitions and special conditions laid on K and h which require some preparatory considerations. For this reason, it is not appropriate to place them in the introduction. Nevertheless, for the reader's convenience, we formulate the main results of this paper at least heuristically as follows. In the following statements, h and h^* are supposed to be in