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# Weakly coupled systems of fully nonlinear parabolic equations in the Heisenberg group

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ABSTRACT

This paper is devoted to viscosity solutions to weakly coupled systems of fully nonlinear parabolic equations in the first Heisenberg group. We extend well-posedness results in the Euclidean space to the Heisenberg group, including the uniqueness and existence of solutions with exponential growth at space infinity under monotonicity and other regularity assumptions on the parabolic operators. In addition, Lipschitz preserving properties of the system are also studied.


## 1. Introduction

In this paper, we discuss uniqueness, existence and Lipschitz regularity of viscosity solutions to a class of weakly coupled systems of fully nonlinear parabolic equations in the first Heisenberg group $\mathbb{H}$:

$$\begin{cases} \partial_t u_i + F_i\left(p, u_1, \ldots, u_m, \nabla_H u_i, (\nabla_H^2 u_i)^\star\right) = 0 & \text{in } \mathbb{H} \times (0, \infty) \text{ for } i = 1, \ldots, m, \quad \text{(a)} \\ u_i(\cdot, 0) = h_i, & \text{in } \mathbb{H} \text{ for } i = 1, \ldots, m, \quad \text{(b)} \end{cases} \tag{1.1}$$

where $F_i : \mathbb{H} \times \mathbb{R}^m \times \mathbb{R}^2 \times \mathcal{S}^2 \to \mathbb{R}$ and $h_i : \mathbb{H} \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are given continuous functions satisfying the assumptions to be elaborated in a moment. Here $\mathcal{S}^k$ stands for the set of all $k \times k$ symmetric matrices. Besides, $\nabla_H u$ and $(\nabla_H^2 u)^\star$ respectively denote the horizontal gradient and the symmetrized horizontal Hessian of a function $u : \mathbb{H} \to \mathbb{R}$; consult Section 2 for precise definitions.

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### 1.1. Background and motivation

Weakly coupled systems of first or second order fully nonlinear equations in the Euclidean space have been studied in a vast literature; see for example [10,15,22,23,21,25] for well-posedness results and applications in optimal control and differential games. More recently, weakly coupled systems involving the $\infty$-Laplacian with a stochastic game interpretation was addressed in [29]. Such nonlinear systems however are far less understood in more general spaces. Motivated by applications of viscosity solution theory in general geodesic spaces [18], in this work we generalize the known results on systems in the Heisenberg group, which is an important example of metric spaces whose geometry is much different from the Euclidean spaces or other Riemannian manifolds.

As for viscosity solution theory on sub-Riemannian manifolds, there has also been extensive study on various nonlinear equations such as the Hamilton–Jacobi equation [28], $p$-Laplace equation with $1 < p \leq \infty$ [3–5,16,6,7] and level set mean curvature flow equation [8,17]; we refer the reader to [27] for a comprehensive introduction of the theory. Noticing that these results are restricted to single equations, we then aim to provide a complement in the case of monotone weakly coupled systems.

Given the background described above, let us state our motivation and discuss main difficulties in our study. As in the Euclidean case, we expect that elliptic and parabolic systems on sub-Riemannian manifolds have broad applications in different fields, especially in image processing; for example, see a model of visual cortex based on the level-set curvature flow equation in the Heisenberg group in [12,11] and see [30] for applications of parabolic systems in the Euclidean space in image analysis. Although it would be desirable to solve more complicated parabolic systems including the level-set mean curvature operator, as a first step, we study in this work a simpler case; our system is fully nonlinear but weakly coupled and nonsingular. Such kind of systems in the Euclidean space has applications in optimal control and differential games [15,23].

Our goal is to lay groundwork for the study of weakly coupled fully nonlinear parabolic systems in the Heisenberg group by establishing uniqueness, existence and Lipschitz regularity of possibly unbounded solutions. Our analysis is not straightforward and the Euclidean theory cannot be directly applied due to the reasons summarized below.

- The equations are fully nonlinear and degenerate parabolic with respect to the Euclidean coordinates. In general one cannot expect smoothing effects of the parabolic operators and thus the solutions have to be interpreted in the viscosity sense. It is of our interest how the coupling structure can affect the viscosity arguments applied in the sub-Riemannian circumstances such as the comparison principle and game-theoretic approximations.
- Uniqueness of unbounded viscosity solutions in an unbounded domain is a difficult issue even in the Euclidean case; see [1] for uniqueness arguments to overcome this difficulty. To establish a comparison principle for our parabolic system, it requires extra work to apply the techniques to the sub-Riemannian setting as well as the system structure.
- It is in general challenging to study Lipschitz continuity of solutions with respect to the metrics of the Heisenberg group, since the group is non-commutative. One cannot directly extend the Lipschitz preserving property in the Euclidean case. The same situation also appears in the system case.

### 1.2. Main results

Let us prepare more notations in order to present our main results. For simplicity, we denote by $\boldsymbol{u}$ the vector-valued function $\mathbb{H} \times [0, \infty) \to \mathbb{R}^m$ with components $u_1, u_2, \ldots, u_m$. For any two vectors $a, b \in \mathbb{R}^m$, we write $a \leq b$ if the components satisfy $a_i \leq b_i$ for all $i = 1, 2, \ldots, m$. Hence, for any $(p, t) \in \mathbb{H} \times [0, \infty)$