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The convexity of the level sets of maximal strictly space-like hypersurfaces defined on 2-dimensional space forms

Peihe Wang[a,*], Xiao Liu[a], Zhonghua Liu[b]

[a] School of Math. Sci., Qufu Normal University, 273165, Qufu Shandong, China
[b] Yishui Campus, Linyi University, 276400, Yishui Shandong, China

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ABSTRACT

For the maximal strictly space-like hypersurface defined on 2-dimensional space forms, we derive some geometrical properties, including the regularity and the strict convexity and the curvature description of the level sets.

1. Introduction

The geometry of solutions to elliptic partial differential equations is a classical subject. The convexity of the solutions or the level sets is one of the interesting aspects. For the convexity of the solutions, for instance, Makar-Limanov [18] and Brascamp–Lieb [4] deduced the convexity of the solution to Poisson equation and the first eigenvalue equation with Dirichlet boundary value problem on bounded convex domain. Caffarelli and Friedman [5] concluded the convexity of the solution to a class of semilinear elliptic equations. Ma and Xu [15] also derived the convexity of the solution to a Hessian equation. For the geometrical properties of the level sets, Alfhors [1] contains the well-known result that level curves of Green function on simply connected convex domain in the plane are the convex Jordan curves. In 1956, Shiffman [23] studied the convexity of the level sets of an immersion minimal annulus in R^3. In 1957, Gabriel [8] proved that the level sets of Green function on a 3-dimensional bounded convex domain are strictly convex. Lewis [11] in 1977 extended Gabriel's result to p-harmonic functions in higher dimensions. Caffarelli–Spruck [6] generalized Lewis's results [11] to

* Corresponding author.
E-mail address: peihewang@hotmail.com (P. Wang).

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a class of semilinear elliptic partial differential equations in 1982. Motivated by the methods proving the convexity of the solutions [5], Korevaar [10] reproved the results of Gabriel and Lewis [8,11] by applying the deformation process and the constant rank theorem. This is a different method to prove the convexity of the level sets and then produced many new results, for instance, Bian–Guan–Ma–Xu [3] and Xu [32] derived the constant rank theorems of the level sets of the solutions to the partial differential equations satisfying some structure conditions.

All the convexity results mentioned above are qualitative. On the quantitative hand, for 2-dimensional harmonic functions and minimal surfaces with convex level curves, Ortel–Schneider [20] and Longinetti [12, 13] proved that the curvature of the level curves attains its minimum on the boundary. Talenti [24] also got the related curvature estimate of the level curves for 2-dimensional harmonic function. Furthermore, Longinetti studied the precise relation between the curvature of the convex level curves and the height of minimal graph in [13]. Obviously, all the quantitative results above are restricted to 2-dimensional cases. The curvature estimate of the level sets of the solution to partial differential equations of higher dimensions is derived by Ma–Ou–Zhang [14] in which they got the Gaussian curvature estimates of the convex level sets of harmonic functions. For the solutions to Poisson equations and a class of semilinear elliptic partial differential equations, the curvature estimate of the level sets has been derived by Wang–Zhang [28] which gave the quantitative description of the convexity result of Caffarelli–Spruck [6], and in the same paper they also studied the properties of the level sets of the minimal graph. In the sequel, Wang [25] got the relation between the curvature of the convex level sets and the height of minimal graph of general dimension which generalized the previous 2-dimensional results by Longinetti [13] and the harmonic functions case due to Ma–Zhang [16].

For the more general Riemannian manifold case, Papadimitrakis [21] proved the convexity of the level curves of harmonic functions on convex rings in the hyperbolic plane via one complex variable tools. Ma–Zhang [17] generalized the results in [21] to general dimensions. Wang-Wang [27] derived the curvature estimate of the steepest descents of harmonic functions on two-dimensional space forms. For the minimal surface defined on the space forms, Wang–Zhao [30], Wang–Zhang [29] and Wang–Qiu–Liu [26] discovered the properties of the level sets which are generalizations of the harmonic functions cases on manifolds [17] and the Euclidean cases of minimal surface [10]. As the basis of this paper, we stated some results in [17] as follows.

Theorem 1.1 (*[17]*)**.** *Let (M^n, g) be a space form with constant sectional curvature, and Ω_0 and Ω_1 be bounded smooth strictly convex domains in M^n, $n \geq 2$ and $\overline{\Omega}_1 \subseteq \Omega_0$. Assume ω solves the following equation*

$$\begin{cases} \triangle\omega = 0 & \text{in} \quad \Omega = \Omega_0 \setminus \overline{\Omega}_1, \\ \omega = 0 & \text{on} \quad \partial\Omega_0, \\ \omega = 1 & \text{on} \quad \partial\Omega_1. \end{cases}$$

Then $\nabla\omega \neq 0$ in Ω and all the level sets of ω are strictly convex with respect to $\nabla\omega$.

On the other hand, as it is well known that the maximal space-like surface in Minkowski space is an object frequently mentioned in mathematics and physics. As an important geometrical surface, it is firstly proposed by Calabi and it is characterized by the zero mean curvature which is similar with minimal surface in Euclidean space. This object is important in Mathematics and Relativity because it can provide sub-manifolds with properties which reflect those of the spacetime. For example, if the weak energy condition is satisfied, then a maximal hypersurface has positive scalar curvature. This fact was important in the initial proof of the positive mass conjecture [7]. Lots of research on it appeared during recent years such as [2,7] and the references therein.

What is interesting to us is the geometry of its level sets. On this aspect, there are several results, for instance, Pyo [22] proved the Shiffman's kind of results of this surface via one complex variable tools,