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Bubbling solutions for an anisotropic planar elliptic problem with exponential nonlinearity[☆]

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A B S T R A C T

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary, we study the following anisotropic elliptic problem

$$\begin{cases} -\nabla\big(a(x)\nabla v\big) = a(x)\big[e^v - s\phi_1 - h(x)\big] & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a(x)$ is a positive smooth function, $h \in C^{0,\alpha}(\overline{\Omega})$, $s > 0$ is a large parameter and ϕ_1 is a positive first eigenfunction of the problem $-\nabla\big(a(x)\nabla\phi\big) = \lambda a(x)\phi$ under Dirichlet boundary condition in Ω. We construct solutions of this problem which exhibit multiple concentration behavior around maximum points of $a(x)\phi_1$ in the domain as $s \to +\infty$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary. This paper deals with the analysis of solutions of the boundary value problem

$$\begin{cases} -\nabla\big(a(x)\nabla v\big) = a(x)\big[e^v - s\phi_1 - h(x)\big] & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $s > 0$ is a large parameter, $h \in C^{0,\alpha}(\overline{\Omega})$ is given, $a(x)$ is a smooth function over $\overline{\Omega}$ and satisfies

$$a_1 \le a(x) \le a_2 \tag{1.2}$$

for some constants $0 < a_1 < a_2 < +\infty$, $\phi_1 > 0$ is an eigenfunction of $-\frac{1}{a(x)}\nabla\big(a(x)\nabla\cdot\big)$ with Dirichlet boundary condition corresponding to the first eigenvalue λ_1. Obviously, if we denote $\rho(x) \in H_0^1(\Omega)$ as a

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unique solution of

$$\begin{cases} -\nabla\big(a(x)\nabla\rho\big) = a(x)h(x) & \text{in} \quad \Omega, \\ \rho = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

then Eq. (1.1) is equivalent to solving for $u = v + \frac{s}{\lambda_1}\phi_1 + \rho$, the problem

$$\begin{cases} -\nabla\big(a(x)\nabla u\big) = a(x)k(x)e^{-t\phi_1}e^u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

where $k(x) = e^{-\rho(x)}$ and $t = s/\lambda_1$.

Eq. (1.1) was motivated by the study of the following elliptic problem of *Ambrosetti–Prodi type* [1]:

$$\begin{cases} -\Delta v = e^v - s\phi_1 - h(x) & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $h \in C^{0,\alpha}(\overline{\Omega})$, $s > 0$ is a large parameter and $\phi_1 > 0$ is an eigenfunction of $-\Delta$ with Dirichlet boundary condition corresponding to the first eigenvalue λ_1. In the early 1980s Lazer and McKenna conjectured that (1.4) has an unbounded number of solutions as $s \to +\infty$ (see [15]). If we set $\rho(x) = (-\Delta)^{-1}h$ in $H_0^1(\Omega)$, then (1.4) is equivalent to solving for $u = v + \frac{s}{\lambda_1}\phi_1 + \rho$, the problem

$$\begin{cases} -\Delta u = k(x)e^{-t\phi_1}e^u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

where $k(x) = e^{-\rho(x)}$ and $t = s/\lambda_1$. When $N = 2$, del Pino and Muñoz [12] gave a *positive* answer to the Lazer–McKenna conjecture for problem (1.4) by constructing *non-simple* bubbling solutions of (1.5) with the following asymptotic behaviors

$$k(x)e^{-t\phi_1}e^{u_t} \rightharpoonup 8\pi\sum_{i=1}^{l} m_i\delta_{\xi_i} \qquad \text{and} \qquad u_t = \sum_{i=1}^{l} m_i G_D(x,\xi_i) + o(1),$$

where $m_i > 1$, ξ_i's are maxima of ϕ_1 and $G_D(x,\xi)$ denotes the Green's function of the problem

$$\begin{cases} -\Delta_x G_D(x,\xi) = 8\pi\delta_\xi(x), & x \in \Omega, \\ G_D(x,\xi) = 0, & x \in \partial\Omega. \end{cases}$$

It is quite surprising that this multiple bubbling phenomenon is in strong opposition to a slightly modified version of Eq. (1.5), namely the Liouville-type equation

$$\begin{cases} -\Delta u = \varepsilon^2 k(x)e^u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.6}$$

where $\varepsilon > 0$ is a small parameter, $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and $k(x) \in \mathcal{C}^2(\overline{\Omega})$ is a nonnegative, not identically zero function. Indeed, it is well known in [4,18,19,23] that if u_ε is a family of solutions of Eq. (1.6) satisfying

$$\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(\Omega)} = +\infty \qquad \text{and} \qquad \lim_{\varepsilon\to 0}\varepsilon^2\int_\Omega k(x)e^{u_\varepsilon}dx = C < +\infty$$

then up to subsequences, $C = 8\pi l$, $l \in \mathbb{N}^*$ and u_ε makes l distinct points *simple* blow up on $\mathcal{S} = \{\xi_1,\ldots,\xi_l\} \subset \Omega$ such that

$$\varepsilon^2 k(x)e^{u_\varepsilon} \rightharpoonup 8\pi\sum_{i=1}^{l}\delta_{\xi_i} \qquad \text{and} \qquad u_\varepsilon = \sum_{i=1}^{l} G_D(x,\xi_i) + o(1).$$