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Positive solutions for double singularly perturbed Schrödinger Maxwell systems

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A B S T R A C T

We show that the number of solutions of a double singularly perturbed Schrödinger Maxwell system on a smooth bounded domain $\Omega \subset \mathbb{R}^3$ depends on the topological properties of the domain. In particular if Ω is non contractible we obtain $\mathrm{cat}(\Omega) + 1$ positive solutions. The result is obtained via Lusternik–Schnirelmann category theory.

1. Introduction

Given real numbers $q > 0$, $\omega > 0$ we consider the following Schrödinger Maxwell stationary system on a smooth bounded domain $\Omega \subset \mathbb{R}^3$.

$$\begin{cases} -\varepsilon^2 \Delta u + u + \omega u v = |u|^{p-2} u & \text{in } \Omega \\ -\varepsilon^2 \Delta v = q u^2 & \text{in } \Omega \\ u, v = 0 \text{ on } \partial\Omega, \ u > 0 \text{ in } \Omega \end{cases} \tag{1}$$

We want to prove that when the parameter ε is sufficiently small, there are many low energy solutions of (1). In particular the number of solutions of (1) is related to the topology of the bounded set Ω.

Schrödinger Maxwell systems recently received considerable attention from the mathematical community [2,6,10,8,9,16,21,24]. For a special case of stationary Schrödinger Maxwell type systems, namely when

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the system is set in $\mathbb{R}^3$, we have an explicit expression for the function v

$$v(u) = \frac{q}{4\pi}\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}dy,$$

and the system is reduced to the following single nonlinear equation:

$$-\Delta u + u + \frac{\omega q}{4\pi}\left(u^2 * \frac{1}{|x|}\right)u = |u|^{p-2}u.$$

This equation is also referred as Schrödinger–Poisson–Slater equation and arises in the Slater approximation in the Hartree–Fock model (see [1,3,16,19,20,23,25,26,29] and the reference therein).

Coming back to the initial system, the singular perturbation of the first equation is widely analyzed in literature — we cite, for instance, [7–9,14] and the reference therein. More recently the mathematical community moved to consider the double perturbed problem [15–18,27,28,30], that is when the singular parameter appears in both equations. In [11] the authors study the evolution of a Schrödinger–Newton system, and it turns out that the double perturbation is needed in order to prove the dynamics of solitary waves when the parameters tend to zero.

Concerning existence of solutions, He [16] studies the following problem

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + uv = f(u) & \text{in } \mathbb{R}^3\\ -\varepsilon^2\Delta v = qu^2 & \text{in } \mathbb{R}^3\\ u>0 \end{cases}$$

where f is a subcritical nonlinearity and V is a suitable potential, while Yang [30] is interested to the system with critical nonlinearity

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + K(x)uv = P(x)g(u) + Q(x)|u|^4u & \text{in } \mathbb{R}^3\\ -\varepsilon^2\Delta v = qu^2 & \text{in } \mathbb{R}^3\\ u>0 \end{cases}$$

where V, K, P, Q are suitable nonhomogeneous potentials. In both cases the existence and multiplicity of solution is given by the properties of the functions V, K, P, Q. The role of the topological properties of the domain on the existence of solution is studied in [15], in which a double perturbed nonlinear system is solved on a Riemannian manifold without boundary. In all these papers a key role is played by the limit problem of the type

$$\begin{cases} -\Delta u + u + \omega uv = |u|^{p-2}u & \text{in } \mathbb{R}^3\\ -\Delta v = qu^2 & \text{in } \mathbb{R}^3\\ u>0 \text{ in } \mathbb{R}^3 \end{cases}$$

and the ground state solutions of this problem will provide a model profile to construct solution for the original problem.

The main difference when the domain has a boundary comes out when looking for the limit problem. In fact, blowing down around a point on the boundary $q \in \partial\Omega$ leads to a problem settled in the half space. The main features of the limit problems in $\mathbb{R}^3$ and in the half space are recalled in Section 2.1 and will be crucial for our result.

Our main result is the following.

Theorem 1. *Let $4 < p < 6$. For ε small enough there exist at least $\operatorname{cat}(\Omega)$ low energy positive solutions of (1). Moreover if Ω is non contractible there is another positive solution with higher energy.*