Nonlinear Analysis 173 (2018) 37–74

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

First-order, stationary mean-field games with congestion

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A R T I C L E   I N F O

Article history:
Received 9 October 2017
Accepted 20 March 2018
Communicated by Enzo Mitidieri

MSC:
35J47
35A01

Keywords:
Mean-field game
Congestion
Calculus of variations

A B S T R A C T

Mean-field games (MFGs) are models for large populations of competing rational agents that seek to optimize a suitable functional. In the case of congestion, this functional takes into account the difficulty of moving in high-density areas. Here, we study stationary MFGs with congestion with quadratic or power-like Hamiltonians. First, using explicit examples, we illustrate two main difficulties: the lack of classical solutions and the existence of areas with vanishing densities. Our main contribution is a new variational formulation for MFGs with congestion. With this formulation, we prove the existence and uniqueness of solutions. Finally, we consider applications to numerical methods.

1. Introduction

Mean-field games (MFGs) is a branch of game theory that studies systems with a large number of competing agents. These games were introduced in [36–38] (see also [39]) and [34,35] motivated by problems arising in population dynamics, mathematical economics, social sciences, and engineering. MFGs have been the focus of intense study in the last few years and substantial progress has been achieved. Congestion problems, which arise in models where the motion of agents in high-density regions is expensive, are a challenging class of MFGs. Many MFGs are determined by a system of a Hamilton–Jacobi equation coupled with a transport or Fokker–Planck equation. In congestion problems, these equations have singularities and, thus, their analysis requires particular care.

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https://doi.org/10.1016/j.na.2018.03.011

Here, we study first-order stationary MFGs with congestion. Our main example is the system

$$\begin{cases} \frac{|P+Du|^\gamma}{\gamma m^\alpha} + V(x) = g(m) + \overline{H} \\ -\operatorname{div}(m^{1-\alpha}|P+Du|^{\gamma-2}(P+Du)) = 0, \end{cases} \tag{1.1}$$

where x takes values on the d-dimensional torus, $\mathbb{T}^d$, and the unknowns are $u, m : \mathbb{T}^d \to \mathbb{R}$ and $\overline{H} \in \mathbb{R}$, with $m \geqslant 0$ and $\int_{\mathbb{T}^d} m\,dx = 1$. Here, $1 \leqslant \alpha \leqslant \gamma < \infty$, $V : \mathbb{T}^d \to \mathbb{R}$, $V \in C^\infty(\mathbb{T}^d)$, and $g : \mathbb{R}^+ \to \mathbb{R}$ with $g(m) = G'(m)$ for some convex function $G : \mathbb{R}_0^+ \to \mathbb{R}$ with $G \in C^\infty(\mathbb{R}^+) \cap C(\mathbb{R}_0^+)$. In particular, the convexity of G means that g is monotonically increasing and is a key factor in the uniqueness of solutions. Here, we do not assume that g is bounded from below and our methods can be used to study the case $g(m) = \ln m$.

Our example MFG is a model in which agents incur in a large cost if they are moving in regions with a high agent density. The constant $-\overline{H}$ is the average cost per unit of time corresponding to the Lagrangian

$$L(x, v, m) = m^\alpha \frac{|v|^{\gamma'}}{\gamma'} + v \cdot P - V(x) + g(m),$$

where $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. More precisely, the typical agent seeks to minimize the long-time average cost,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(m^\alpha \frac{|\dot{\mathbf{x}}(s)|^{\gamma'}}{\gamma'} + \dot{\mathbf{x}}(s) \cdot P - V(\mathbf{x}(s)) + g(m(\mathbf{x}(s))) \right) ds.$$

Due to this optimization process, agents avoid moving in high-density regions. Further, because g is increasing, agents prefer to remain in low-density regions rather than in high-density regions. The case where g is non-increasing was examined in [11,13], for example. Finally, we observe that P determines the preferred direction of motion.

In the stationary case, the theory for second-order MFGs without singularities is well understood. For example, [23,26–28,44] address the existence of classical solutions and weak solutions were examined in [7]. Some models with a different class of singularities were examined in [24] and in [12]. In dimension one, a characterization of solutions for stationary MFGs was developed in [22] (including non-monotone MFGs) and, in the case of congestion, in [21,43]. The theory of weak solutions was considered in [18], where a general existence result was proven using a monotonicity argument. The monotonicity structure that many MFGs enjoy has important applications to numerical methods, see [4]. A review of MFG models can be found in [29] and a survey of regularity results in [25].

The congestion problem was first introduced in [39] where a uniqueness condition was established. Next, the existence problem for stationary MFGs with congestion, positive viscosity, and a quadratic Hamiltonian was proved in [20]. Subsequently, this problem was examined in more generality in [16]. The time-dependent case was considered in [30] (classical solutions) and [31] (weak solutions). Later, [3] examined weak solutions for time-dependent problems.

Apart from the results in [18], the one-dimensional examples in [21,43], and the radial cases in [17], little is known about first-order MFGs with congestion. The critical difficulties stem from two issues: first-order Hamilton–Jacobi equations provide little a priori regularity; second, because the transport equation is a first-order equation, we cannot use Harnack-type results and, thus, we cannot bound m from below with a positive constant. Indeed, as we show in Section 2.1, m can vanish. In many MFG problems, the regularity follows from a priori bounds that combine both equations in (1.1). Here, with standard methods, we can only get relatively weak bounds, see Remark 3.6. For example, if $0 < \alpha \leqslant 1$, then there exists a constant, C, such that for any regular enough solution of (1.1), we have

$$\int_{\mathbb{T}^d} \left[\left(\frac{|P+Du|^\gamma}{m^\alpha} \right) (1+m) + (m-1) g(m) \right] dx \leqslant C.$$

In Section 3, we examine a priori bounds for a class of MFGs that generalize (1.1).