Nonlinear Analysis 173 (2018) 146–153

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Generalized Gagliardo–Nirenberg inequalities using Lorentz spaces, BMO, Hölder spaces and fractional Sobolev spaces

Nguyen Anh Dao[a,*], Jesus Ildefonso Díaz[b], Quoc-Hung Nguyen[c]

[a] *Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Viet Nam*
[b] *Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, 28040 Madrid, Spain*
[c] *Scuola Normale Superiore, Centro Ennio de Giorgi, Piazza dei Cavalieri 3, I-56100 Pisa, Italy*

ARTICLE INFO

Article history:
Received 26 September 2017
Accepted 1 April 2018
Communicated by Enzo Mitidieri

Keywords:
Gagliardo–Nirenberg's inequality
Lorentz spaces
BMO
Lipschitz space
Fractional Sobolev space

ABSTRACT

The main purpose of this paper is to prove some generalized Gagliardo–Nirenberg interpolation inequalities involving the Lorentz spaces $L^{p,\alpha}$, BMO and the fractional Sobolev spaces $W^{s,p}$, including also $\dot{\mathcal{C}}^{\eta}$ Hölder spaces. Although some of the results can be alternatively obtained by using interpolation spaces (specifically, the reiteration theorem), the precise form of the inequalities stated here appears to be novel and, moreover, the proofs given in the present paper are self-contained (save for the use of the John–Nirenberg inequality for the BMO result) in contrast to the other mentioned approach. The use of $\dot{\mathcal{C}}^{\eta}$ Hölder spaces in such Gagliardo–Nirenberg inequalities seems to be new in the literature.

1. Introduction

The main purpose of this paper is to prove some generalized Gagliardo–Nirenberg interpolation inequalities involving the Lorentz spaces $L^{p,\alpha}$, BMO, and the fractional Sobolev spaces $W^{s,p}$, including also $\dot{\mathcal{C}}^{\eta}$ Hölder spaces.

It is well known that the Gagliardo–Nirenberg inequality plays an important role in the analysis of PDEs, see e.g. [10,8,9,1,11,7,3] and the references therein. Thus, any possible improvement of this one could be relevant for many purposes. First of all, let us recall some previous results involving the Gagliardo–Nirenberg inequalities that we shall improve later:

For any $1 \leq q < p < \infty$, the following interpolation inequality holds (see Nirenberg [10])

$$\|f\|_{L^p(\mathbb{R}^n)} \leq c\|f\|^{\theta}_{L^q(\mathbb{R}^n)}\|f\|^{1-\theta}_{\dot{H}^s(\mathbb{R}^n)}, \quad \frac{1}{p} = \frac{\theta}{q} + (1-\theta)(\frac{1}{2} - \frac{s}{n}). \tag{1.1}$$

* Corresponding author.
E-mail addresses: daonguyenanh@tdt.edu.vn (N.A. Dao), ildefonso.diaz@mat.ucm.es (J.I. Díaz), quoc-hung.nguyen@sns.it (Q.-H. Nguyen).

https://doi.org/10.1016/j.na.2018.04.001

More recently, in [9], McCormick et al. proved a stronger version of (1.1) involving the weak L^q space (denoted as $L^{q,\infty}$) as follows:

$$\|f\|_{L^p(\mathbb{R}^n)} \le c\|f\|^{\theta}_{L^{q,\infty}(\mathbb{R}^n)}\|f\|^{1-\theta}_{\dot{H}^s(\mathbb{R}^n)}. \tag{1.2}$$

Concerning the critical case $s = n/2$, McCormick et al. [9] obtained

$$\|f\|_{L^p(\mathbb{R}^n)} \le C\|f\|^{q/p}_{L^{q,\infty}(\mathbb{R}^n)}\|f\|^{1-q/p}_{BMO(\mathbb{R}^n)}. \tag{1.3}$$

Note that (1.3) is better than (1.2) since $\|f\|_{BMO(\mathbb{R}^n)} \le c\|f\|_{\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)}$. Furthermore, they also showed a stronger version of inequality (1.3) for the norm $\|f\|_{L^{p,1}(\mathbb{R}^n)}$ instead of $\|f\|_{L^p(\mathbb{R}^n)}$ when $q > 1$.

Another version of (1.1), in the critical case, was proved by Kozono and Wadade [8] (see also [1]):

$$\|f\|_{L^p(\mathbb{R}^n)} \le c\|f\|^{\frac{q}{p}}_{L^q(\mathbb{R}^n)}\|f\|^{1-\frac{q}{p}}_{\dot{H}^{\frac{n}{r},r}(\mathbb{R}^n)}, \tag{1.4}$$

for any $1 \le q < p < \infty$, and for $1 < r < \infty$.

In this paper, we enhance the results of McCormick et al., [9]. We shall prove a stronger version of (1.2)

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \le c\|f\|^{\theta}_{L^{q,\infty}(\mathbb{R}^n)}\|f\|^{1-\theta}_{\dot{H}^s(\mathbb{R}^n)}. \tag{1.5}$$

In fact, we shall prove an interpolation inequality which implies (1.5) (see Theorem 2.1 below). After that, we shall prove that (1.3) holds for $\|f\|_{L^{p,\alpha}}$ instead of $\|f\|_{L^p}$, for any $\alpha > 0$. Finally, we shall study the Gagliardo–Nirenberg type inequality for the case $sp > n$ of the fractional Sobolev space $W^{s,p}$, and also the Lipschitz and Hölder continuous space. Let us point out that although some of the results can be alternatively obtained by using interpolation spaces (specifically, the reiteration theorem), the precise forms of the inequalities stated here appear to be novel and, moreover, the proofs given in the present paper are self-contained (save for the use of the John–Nirenberg inequality for the BMO result, which will be recalled later) in contrast to the other mentioned approach.

For the reader convenience, we recall here the definition of the functional spaces that we use throughout this paper. We define

$$\|g\|_{L^{q,\alpha}(\mathbb{R}^n)} := \begin{cases} \left(q\displaystyle\int_0^\infty \left(\lambda^q|\{x\in\mathbb{R}^n : |g(x)|>\lambda\}|\right)^{\frac{\alpha}{q}}\frac{d\lambda}{\lambda}\right)^{1/\alpha} & \text{if } \alpha<\infty,\\ \sup\limits_{\lambda>0}\lambda\left(|\{x\in\mathbb{R}^n : |g(x)|>\lambda\}|\right)^{1/q} & \text{if } \alpha=\infty.\end{cases}$$

The Lorentz spaces $L^{q,\alpha}(\mathbb{R}^n)$ includes all measurable functions $g : \mathbb{R}^n \to \mathbb{R}$ such that $\|g\|_{L^{q,\alpha}(\mathbb{R}^n)} < \infty$. For a definition of Lorentz spaces using rearrangement techniques see [12].

On the other hand, we recall that the space $\dot{H}^{s,r}(\mathbb{R}^n)$, the homogeneous Sobolev space, is defined by

$$\dot{H}^{s,r}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|(-\Delta)^{\frac{s}{2}}f\|_{L^r} < \infty\}.$$

In particular, we shall denote $\dot{H}^s(\mathbb{R}^n) = \dot{H}^{s,2}(\mathbb{R}^n)$ (see, for instance, [5]).

We denote the space of Lipschitz (or Hölder) continuous functions of order $\eta \in (0, 1]$ on $\mathbb{R}^n$ by $\dot{C}^\eta(\mathbb{R}^n)$: i.e. functions f such that

$$\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\eta} < \infty.$$

It is useful to introduce the notation

$$\|f\|_{\dot{C}^\eta(\mathbb{R}^n)} = \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\eta}$$

(see. e.g. section 6.3 of [5]).