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Homogenization of a class of singular elliptic problems in perforated domains

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A B S T R A C T

In this work we study the asymptotic behavior of a class of quasilinear elliptic problems posed in a domain perforated by ε-periodic holes of size ε. The quasilinear equations present a nonlinear singular lower order term $f\zeta(u_\varepsilon)$, where u_ε is the solution of the problem at ε-level, ζ is a continuous function singular in zero and f a function whose summability depends on the growth of ζ near its singularity. We prescribe a nonlinear Robin condition on the boundary of the holes contained in Ω and a homogeneous Dirichlet condition on the exterior boundary. The particular case of a Neumann boundary condition on the holes is already new.

The main tool in the homogenization process consists in proving a suitable convergence result, which shows that the gradient of u_ε behaves like that of the solution of a suitable linear problem associated with a weak cluster point of the sequence $\{u_\varepsilon\}$, as $\varepsilon \to 0$. This allows us not only to pass to the limit in the quasilinear term, but also to study the singular term near its singularity, via an accurate a priori estimate. We also get a corrector result for our problem.

The main novelty of this work is that for the first time the unfolding method is used to treat a singular term as $f\zeta(u_\varepsilon)$. This plays an essential role in order to get an almost everywhere convergence of the solution u_ε, needed in the study the asymptotic behavior of the problem.

1. Introduction

In this paper we deal with the homogenization of a class of quasilinear elliptic problems with singular lower order terms posed in periodically perforated domains. In our study, we use the periodic unfolding method, originally introduced in [13] and [14]. The perforated domain Ω_ε^* is obtained by removing from a

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connected bounded open set Ω of $\mathbb{R}^N$, $N \geq 2$, a set of ε-periodic holes of size ε. The boundary of Ω_ε^* is decomposed into Γ_1^ε and Γ_0^ε, which denote the boundary of the holes well contained in Ω and the remaining exterior boundary, respectively (see Section 2 for details). We prescribe a nonlinear Robin condition on Γ_1^ε and a homogeneous Dirichlet condition on Γ_0^ε.

More precisely, we study the asymptotic behavior, as ε goes to zero, of the following problem:

$$\begin{cases} -div(A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon) = f\zeta(u_\varepsilon) & \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ (A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon)\nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \end{cases}$$

where ν is the unit outward normal to the holes.

The oscillating coefficients' matrix field A^ε in the quasilinear diffusion term is defined by $A^\varepsilon(x,t) = A\left(\frac{x}{\varepsilon}, t\right)$, where the matrix field A is uniformly elliptic, bounded, periodic in the first variable and Carathéodory. The nonlinear real function $\zeta(s)$ is nonnegative and singular at $s = 0$, while f is a nonnegative datum whose summability depends on the growth of ζ near its singularity. Concerning the Robin boundary condition, $\rho_\varepsilon(x) = \rho\left(\frac{x}{\varepsilon}\right)$ where the function ρ is assumed to be periodic, nonnegative and bounded on ∂T, the nonlinear boundary term h is an increasing and continuously differentiable function whose derivative satisfies suitable growth assumptions, and $g_\varepsilon(x) = \varepsilon g\left(\frac{x}{\varepsilon}\right)$, where g is a periodic nonnegative function with prescribed summability.

From the physical point of view, the quasilinear diffusion term describes the behavior of some materials, like glass or wood, in which the heat diffusion depends on the temperature (see [5] for more details). A source term depending on the solution itself and becoming infinite when the solution vanishes may model an electrical conductor, where each point becomes a source of heat as a current flows in it (see [23, Section 3]). Moreover, nonlinear Robin boundary conditions describe certain chemical reactions at the boundaries of perforations (e.g. [21]).

The existence and uniqueness of the weak solution of the problem, for every fixed ε, have already been proved by the authors in [24]. Uniform a priori estimates of the solution u_ε are obtained by adapting to our case some arguments introduced in [24]. Let us mention that the third bound (cf. Proposition 4.4) provides an estimate of the integral of the singular term close to the singular set $\{u_\varepsilon = 0\}$, in terms of the quasilinear one.

In the homogenization process we have to pass to the limit in the quasilinear term, in the singular one and in the nonlinear Robin condition. In order to study the quasilinear term, we prove a crucial convergence, given by Theorem 5.5, which is the main tool when proving our results. It shows that the gradient of u_ε behaves like that of the solution of a suitable linear problem associated with a weak cluster point of the sequence $\{u_\varepsilon\}$, as $\varepsilon \to 0$. This idea was originally introduced in [3] (see also [4]) where some nonlinear problems with quadratic growth are considered. The main difference with respect to [3] consists in the choice of the test functions used in the proof. Indeed, in the quadratic growth case the authors take exponential test functions, while here we have to use appropriate test functions that allow us to treat the singular term. To construct these functions, taking into account the homogenization results of [10], we adapt to our needs some techniques from [23].

As far as it concerns the singular term, as done in [23] and [24], we split it into the sum of two integrals: one on the set where the solution is close to the singularity and one where is it far from it, which results not singular. Near the singularity, we make use of the estimate given by Proposition 4.4. In this way, we shift the study of the singular term to that of the quasilinear one, for which we can use the previous result.

Let us point out that for $h \equiv 0$ we obtain, as a particular case, homogenization and corrector results for the problem when a (homogeneous or not) Neumann boundary condition on Γ_1^ε is prescribed, which are already new in the literature.