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Variable Lorentz estimate for nonlinear elliptic equations with partially regular nonlinearities

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A B S T R A C T

We prove global Calderón–Zygmund type estimate in Lorentz spaces for variable power of the gradients to weak solution of nonlinear elliptic equations in a non-smooth domain. We mainly assume that the nonlinearities are merely measurable in one of the spatial variables and have sufficiently small BMO semi-norm in the other variables, the boundary of domain belongs to Reifenberg flatness, and the variable exponents $p(x)$ satisfy log-Hölder continuity.

1. Introduction

Throughout this paper, let Ω be a bounded domain $\mathbb{R}^n (n \geq 2)$ with a rough boundary specified later. Suppose that $\mathbf{F} = (f^1, f^2, \ldots, f^n)$ is a given vector-valued measurable function, and $\mathbf{a} = \mathbf{a}(\xi, x) : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a Carathéodory vector valued function which is measurable in $x \in \Omega$ for each $\xi \in \mathbb{R}^n$ and Lipschitz continuous in $\xi \in \mathbb{R}^n$ for each $x \in \Omega$. The aim of this article is to study a global Lorentz estimate for variable power of the gradients to weak solution of the Dirichlet problem for nonlinear elliptic equations:

$$\begin{cases} \text{div } \mathbf{a}(Du, x) = \text{ div } \mathbf{F}, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

under very weak assumptions that the nonlinearities $\mathbf{a}(\xi, x)$ are merely *small partially BMO (Bounded Mean Oscillation)* semi-norm in the spatial variables and $\partial\Omega$ is Reifenberg flat. The weak solution of (1.1)

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is understood in the usual sense: for $u \in W_0^{1,2}(\Omega)$ it holds

$$\int_\Omega \langle \mathbf{a}(Du,x), D\varphi \rangle dx = \int_\Omega \langle \mathbf{F}, D\varphi \rangle dx, \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{1.2}$$

To ensure solvability in $L^2(\Omega)$ of (1.1), we need to impose a structural assumption with ellipticity and growth: there exist two constants $0 < \nu \le \Lambda < \infty$ such that

$$\begin{cases} \langle D_\xi \mathbf{a}(\xi,x)\,\eta,\eta\rangle \ge \nu|\eta|^2, \\ |\mathbf{a}(\xi,x)| + |\xi|\,|D_\xi \mathbf{a}(\xi,x)| \le \Lambda|\xi| \end{cases} \tag{1.3}$$

for a. e. $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^n$, where D_ξ denotes the differentiation in $\xi \in \mathbb{R}^n$, and $\langle \cdot,\cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. By (1.3) it is clear to check that

$$\begin{cases} \mathbf{a}(0,x) = 0, \\ \nu|\xi-\eta|^2 \le \langle \mathbf{a}(\xi,x) - \mathbf{a}(\eta,x), \xi-\eta\rangle. \end{cases} \tag{1.4}$$

Therefore, by the usual Minty–Browder argument there exists a unique weak solution $u \in W_0^{1,2}(\Omega)$ of (1.1) with the following L^2 estimate

$$\|Du\|_{L^2(\Omega)} \le c\|\mathbf{F}\|_{L^2(\Omega)},$$

where c is a constant independent of $u, \mathbf{F}$ and Ω.

The Calderón–Zygmund theory concerned partial differential equations with partially regular coefficient assumptions has been getting great attention. For the case of linear PDEs, this was first introduced by Kim and Krylov in [19], and later employed by Dong and Kim in [14,15,17] and by Byun and Wang in [11] in the study of Calderón–Zygmund theory to divergence and nondivergence linear elliptic and parabolic equations/systems with partially VMO or small partially BMO coefficients. It has actually proved to be a sort of minimal regular requirement imposed on the leading coefficients for elliptic and parabolic operators to ensure a satisfactory Calderón–Zygmund theory for all $p > 1$. Indeed, this was verified due to a famous counterexample by Ural'tseva [29], who constructed an example of an equation in $\mathbb{R}^d\,(d \ge 3)$ with the coefficients depending only on the first two coordinates so that we reached that there is no unique solvability in Sobolev spaces $W^{1,p}$ for any $p > 1$. It is worth noting that Byun–Wang [11] and Byun–Palagachev [9] considered linear elliptic equations with partially BMO coefficients and obtained the L^p-estimate, weighted L^p-estimate, respectively. We are now interested in nonlinear elliptic equations with partially regular nonlinearities in the spatial variables since those are related to nonlinear problems in medium composition materials. We would particularly like to point out that the study of this article was inspired by two recent progresses from Byun et al.'s papers. Byun, Ok and Wang [8] obtained a global $L^{p(x)}$ estimate to the Dirichlet problem of divergence linear elliptic system in Reifenberg domain with *partially BMO coefficients* and *log-Hölder continuity* $p(x)$, who showed that

$$\mathbf{F} \in L^{p(x)}(\Omega, \mathbb{R}^n) \Rightarrow Du \in L^{p(x)}(\Omega, \mathbb{R}^n), \qquad p(x) \ge 2.$$

On the other hand, Byun and Kim [7] also established global L^p theory to divergence nonlinear elliptic equations (1.1) with measurable nonlinearities, which means that

$$\mathbf{F} \in L^p(\Omega) \Rightarrow Du \in L^p(\Omega), \quad 2 \le p < \infty$$

for weak solution $u \in W_0^{1,2}(\Omega)$ of the Dirichlet problems (1.1). Therefore, a refined natural outgrowth of the above-mentioned two papers leads to our consideration in the framework of variable Lorentz spaces.

As we know, Lorentz spaces are a two-parameter scale of the Lebesgue spaces by refining Lebesgue spaces in the fashion of second index. Recently there were a large number of literatures on the topic concerning